Norwegian University of Science and Technology
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Introductory course in linear algebra and differential equations
Autumn 2019

Solutions to exercise set 7

$\boxed{1}$ We start by finding eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{bmatrix}.$$

Since A is symmetric, we know that we can find an orthogonal diagonalization. The characteristic polynomial is

$$p(\lambda) = (-0.01 - \lambda)^2 - 0.01^2 = \lambda(\lambda + 0.02),$$

so that the eigenvalues are $\lambda_1 = -0.02$ and $\lambda_2 = 0$. Since

$$A - (-0.02I) = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

we can choose $v_1 = (1, -1)/\sqrt{2}$ (we could also have written v_2 directly, since we know it is perpendicular to v_1). Further, since

$$A - 0I = \begin{bmatrix} -0.01 & 0.01 \\ 0.01 & -0.01 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}$$

we can choose $v_2 = (1, 1)/\sqrt{2}$. The matrix A can therefore be diagonalized as

$$A = PDP^T, \quad P = \begin{bmatrix} v_1 v_2 \end{bmatrix}, D = \operatorname{diag}(-0.02, 0).$$

We have now

$$\begin{aligned} e^{At} &= Pe^{Dt}P^T \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} e^{-0.02t} & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 + e^{-0.02t} & 1 - e^{-0.02t} \\ 1 - e^{-0.02t} & 1 + e^{-0.02t} \end{bmatrix}, \end{aligned}$$

so that the solution is given by

$$\begin{aligned} x(t) &= \frac{1}{2}\begin{bmatrix} 1 + e^{-0.02t} & 1 - e^{-0.02t} \\ 1 - e^{-0.02t} & 1 + e^{-0.02t} \end{bmatrix}\begin{bmatrix} 100 \\ 0 \end{bmatrix} \\ &= 50\begin{bmatrix} 1 + e^{-0.02t} \\ 1 - e^{-0.02t} \end{bmatrix}. \end{aligned}$$

There are 25 grams of salt in the second tank when

$$50(1 - e^{-0.02t}) = 25,$$

which yields $t = 50\log(2) \approx 35$ seconds.

$\boxed{2}$ We find

$$A - I = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix},$$

so that the eigenvalue 1 has one corresponding eigenvector $v_1 = (1, 1, 0)$. For the eigenvalue 3 we have

$$A - 3I = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here $x_2 = s$ and $x_3 = t$ are free variables, and $x_1 = t - s$. The vectors in the null space is thus on the form

$$x = (t - s, s, t) = s(-1, 1, 0) + t(1, 0, 1),$$

and we conveniently pick the eigenvectors $v_2 = (-1, 1, 0)$ og $v_3 = (1, 0, 1)$ for the eigenvalue 3.

The general solution of the differential equation is thus

$$x(t) = a_1 e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + a_2 e^{3t} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + a_3 e^{3t} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

To find the right coefficients for our initial datum, we solve the system of equations

$$\begin{bmatrix} 1 & -1 & 1 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

Gaussian elimination gives us

$$\begin{bmatrix} 1 & -1 & 1 & 1 \\ 1 & 1 & 0 & 2 \\ 0 & 0 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix}.$$

By backward substitution we find $a_1 = a_2 = a_3 = 1$, so that the solution of the initial value problem is

$$x(t) = e^t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + e^{3t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t \\ e^t + e^{3t} \\ e^{3t} \end{bmatrix}.$$