NTNU
Norwegian University of Science and Technology

Department of Mathematical Sciences

Examination paper for **TMA4110 Mathematics 3 - Linear algebra**

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Examination date: December 1st 2021

Examination time (from–to): 09:00–13:00

Permitted examination support material: C: Specified printed and hand-written support material is allowed. A specific basic calculator is allowed (Casio fx-82ES PLUS, Casio fx-82EX, Citizen SR-270X, Citizen SR-270X College, Hewlett Packard HP30S).

Other information:
The exam consists of 10 subproblems. All subproblems are given equal weight. Give reasons for all answers. This year we specify that NO printed or handwritten support material is allowed.

Language: English

Number of pages: 2

Number of pages enclosed: 0

Checked by:

Informasjon om trykking av eksamensoppgave	
Originalen er:	
1-sidig ☐	**2-sidig** ☒
sort/hvit ☒	**farger** ☐
skal ha flervalgskjema ☐	

Date Signature

Problem 1 Determine all roots of $p(z) = z^4 + 16$ and plot the roots in the complex plane. Factorize $p(z)$ in terms of linear factors.

Problem 2 Consider the three points

$$(0, -3), (1, -1) \text{ og } (2, 5)$$

in $\mathbb{R}^2$.
Find the polynomial $p(x) = ax^2 + bx + c$ of degree two that passes through all these points and apply the method of least squares to determine the polynomial $q(x) = dx + e$ of degree one that is the best fit for the three points.

Problem 3 Let A be the 3×3-matrix

$$A = \begin{bmatrix} a-1 & 4 & 2 \\ 0 & a & 1 \\ 0 & 6 & a+1 \end{bmatrix}.$$

a) Determine all real numbers a such that $\det A \neq 0$. Determine the dimension of $\operatorname{Col} A$ (the column space of A) for all values of a.

b) Depending on a, find all real numbers b such that the system

$$A \cdot \mathbf{x} = \begin{bmatrix} 6 \\ b \\ 8 \end{bmatrix}$$

has a solution.

Problem 4 Let

$$A = \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & -1/2 \end{bmatrix}.$$

Let $T \colon \mathbb{R}^2 \to \mathbb{R}^2$ be the linear transformation given by $T(\mathbf{x}) = A\mathbf{x}$.

a) Give a geometric interpretation of the action of this linear transformation, and calculate A^{2021}.

b) Determine the eigenvalues of A and give a geometric interpretation of the fact that they are not real numbers.

Problem 5 Let $V = \mathcal{C}[0,1]$ be the vector space of all continuous functions $f\colon [0,1] \to \mathbb{R}$, and consider the subspace $U = \mathrm{Sp}\{1, x\}$. Let $\langle f, g\rangle = \int_0^1 f(x)g(x)dx$. This gives an inner product for $V = \mathcal{C}[0,1]$.

a) Determine an orthogonal basis for U.

b) Let $h(x) = e^x$. Compute $\mathrm{Proj}_U(h(x))$.
Hint: $((x-1)e^x)' = xe^x$.

Problem 6 Find a real 2×2-matrix A that is associated with the system of differential equations $\mathbf{y}'(t) = A\mathbf{y}(t)$ having solutions

$$e^{2t}\begin{bmatrix}1\\0\end{bmatrix} \text{ og } e^{2t}\left(\begin{bmatrix}t\\0\end{bmatrix} + \begin{bmatrix}0\\1\end{bmatrix}\right).$$

Problem 7

Let

$$B = \begin{bmatrix}1 & 0 & -2\\0 & 2 & -1\\3 & 0 & -4\end{bmatrix},$$

Find an invertible 3×3-matrix A satisfying

$$3A = A^2 - AB$$

and explain why there does not exist an invertible 3×3-matrix A such that

$$2A = A^2 - AB.$$