Problems for First day

1. Verify that both

$$y_1(x) = 5e^x \text{ and } y_2(x) = 3e^x$$

solves

$$y'(x) = y(x)$$

Can you find more solutions? Sketch the solutions in the xy-plane.

2. Verify that both

$$y_1(t) = 3e^{-2t} \text{ and } y_2(t) = -2e^{3t}$$

are solutions of

$$y'' - y' - 6y = 0$$

Is $y(t) = 10y_1(t) + 2y_2(t)$ also a solution?

3. Determine for which real values $a \in \mathbb{R}$ the function $y(t) = \arctan(t)$ satisfy the differential equation

$$(t^2 + 1)y''(t) + aty'(t) = 0$$

4. Find the equilibrium points and classify them as either stable or unstable for the ODE

$$x'(t) = f(x)$$

where

a) $f(x) = x\left(\frac{1}{2} - e^{-|x|}\right)$

b) $f(x) = |x| - 1$

c) $f(x) = \sin(x)$

5. Solve the initial value problem

$$y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}, \quad y\left(\frac{\pi}{2}\right) = 0$$

by using integrating factor.

6. Use separation of variables to solve the initial value problem

$$y' - t^2\sqrt{y} = 0, \quad y(0) = 1.$$

7. Solve the following ODEs

a) $y'(x) = \frac{3y-1}{x}$

b) $y'(x) + \frac{2y}{x} = \frac{1}{x^2}$

8. Find an equation for a curve going through the point $(2, 3)$ and which has slope $\frac{2x}{1+y^2}$.

Solutions for First day

1. We find $y_1'(t)$ and $y_2'(t)$ and check if they satisfy the equation.

$$y_1'(x) = 5e^x$$

and we see that

$$y_1'(x) = 5e^x = y_1(x)$$

so y_1 is a solution of

$$y' = y$$

We also have

$$y_2'(x) = 3e^x = y_2(x)$$

so this is also a solution. In general, we have for every constant $C \in \mathbb{R}$ that $y = Ce^x$ is a solution of the equation.

2. We start by finding the first and second derivative of y_1 and y_2,

$$\begin{aligned} y_1'(t) &= -6e^{-2t} \\ y_1''(t) &= 12e^{-2t} \end{aligned}$$

and

$$\begin{aligned} y_2'(t) &= -6e^{3t} \\ y_2''(t) &= -18e^{3t} \end{aligned}$$

Now, we see that

$$\begin{aligned} y_1''(t) - y_1'(t) - 6y_1(t) &= 12e^{-2t} + 6e^{-2t} - 18e^{-2t} \\ &= e^{-2t}(12 + 6 - 18) \\ &= e^{-2t} \cdot 0 \\ &= 0 \end{aligned}$$

so y_1 solves the equation. Also

$$\begin{aligned} y_2''(t) - y_2'(t) - 6y_2(t) &= -18e^{3t} + 6e^{3t} + 12e^{3t} \\ &= e^{-2t}(-18 + 6 + 12) \\ &= e^{-2t} \cdot 0 \\ &= 0 \end{aligned}$$

so y_2 also solves the equation. Now, we can see that for any constants c_1 and c_2 we have that $y(t) = c_1y_1(t) + c_2y_2(t)$ gives us

$$\begin{aligned} y''(t) - y'(t) - 6y &= (c_1y_1(t) + c_2y_2(t))'' - (c_1y_1(t) + c_2y_2(t))' - 6(c_1y_1(t) + c_2y_2(t)) \\ &= c_1(y_1''(t) - y_1'(t) - 6y_1(t)) + c_2(y_2''(t) - y_2'(t) - 6y_2(t)) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

so in particular, if $c_1 = 10$ and $c_2 = 2$, we have that $y(t)$ is a solution of the ODE.

3. By differentiation, we get that

$$y'(t) = \frac{1}{1+t^2}, \quad y''(t) = -\frac{2t}{(1+t^2)^2},$$

so if y solves the differential equation, we get that

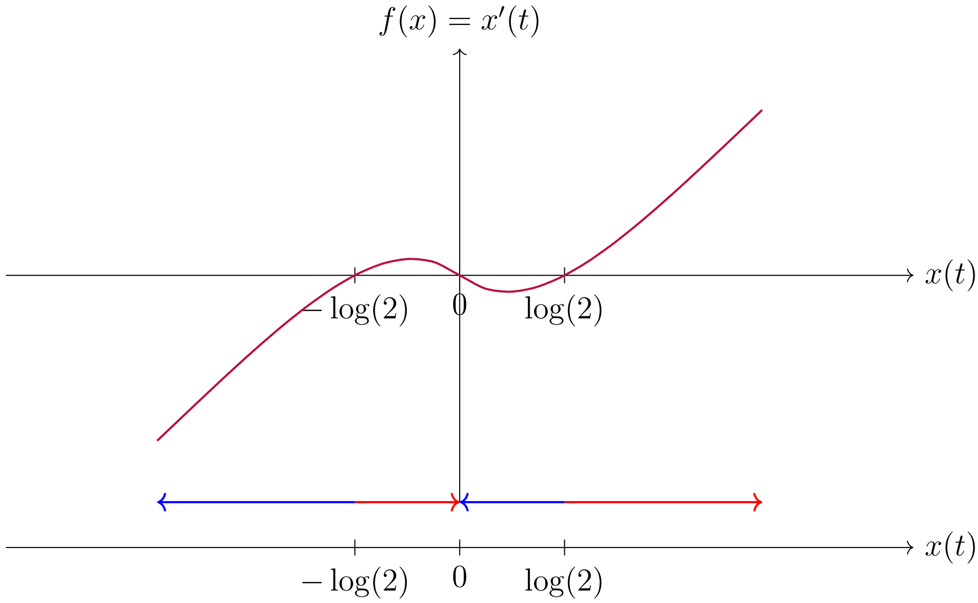
$$(t^2+1)\left(\frac{-2t}{(1+t^2)^2}\right) + at\frac{1}{1+t^2} = 0.$$

Thus,

$$\frac{-2t + at}{1 + t^2} = 0,$$

For this to be satisfied for all t, we see that $a = 2$.

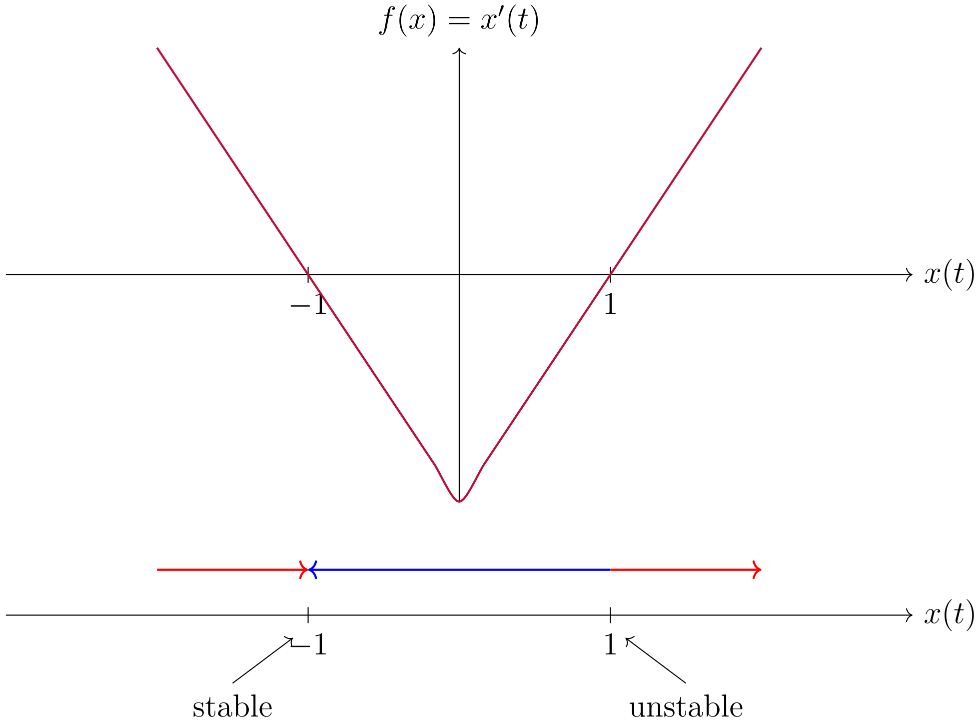
4.a) The function is zero in the points, $x = 0$ and $x = \pm \ln(2)$. These are the equilibrium points of the differential equation. Below is a drawing of f, and the qualitative behavior of the solutions to the differential equation.



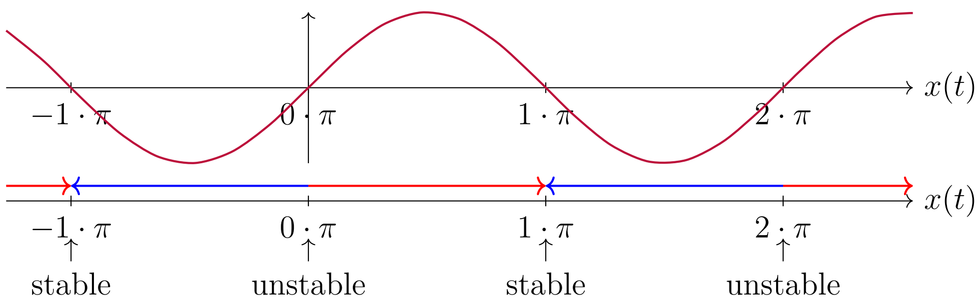
- If our initial data is on the interval $(-\infty, -\ln(2))$ the solution will go to ∞ as $t \to \infty$.
- On the interval $(-\ln(2), 0)$, the solution tends to the equilibrium point 0.
- On the interval $(0, \ln(2))$, the solution also tends to the equilibrium point 0.
- On the interval $(\ln(2), \infty)$, the solution tends to $+\infty$.

From the list above, we observe that the equilibrium point 0 is stable, while $\pm \ln(2)$ are unstable.

b)



c)



5. Let us for the sake of being lazy, assume that $t > 0$, then the integrating factor is

$$e^{F(t)} = e^{\int 2/t dt} = e^{2\ln(t)} = t^2$$

and we have

$$\begin{aligned} t^2y' + t^2\frac{2}{t}y &= t^2\frac{\cos(t)}{t^2} \\ \frac{d}{dt}\left(t^2y(t)\right) &= \cos(t) \\ t^2y(t) &= \int \cos(t)dt \\ t^2y(t) &= \sin(t) + C \\ y(t) &= \frac{\sin(t)+C}{t^2} \end{aligned}$$

since, we have the condition $y(\pi/2) = 0$, we need to have $C = -1$, so the solution of the initial value problem is

$$y(t) = \frac{\sin(t) - 1}{t^2}$$

Do note that this is only a solution for $t > 0$.

6. We write the equation as

$$y^{-1/2}y' = t^2,$$

and we proceed to integrate it

$$\int \frac{y'(t)}{\sqrt{y(t)}}dt = \int t^2 dt = \frac{1}{3}t^3 + C$$

On the left hand side we use the substitution $u = y(t)$, which gives us

$$du = y'(t)dt$$

and therefore the left hand side is

$$\int \frac{y'(t)}{\sqrt{y(t)}}dt = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u} = 2y^{1/2}.$$

Thus, we have

$$2y^{1/2} = \frac{1}{3}t^3 + C$$

and therefore

$$y(t) = \left(\frac{C}{2} + \frac{1}{6}t^3\right)^2.$$

With the initial data we have

$$y(t) = \left(1 + \frac{1}{6}t^3\right)^2$$

7. a) We can rewrite the equation as

$$y' - \frac{3}{x}y = -\frac{1}{x}$$

The integrating factor is given by

$$e^{F(x)} = e^{\int -\frac{3}{x}dx} = e^{-3\ln|x|}.$$

This gives us

$$\frac{d}{dc}\left(e^{-3\ln|x|}y(x)\right) = -\frac{1}{x}e^{-3\ln|x|}.$$

Taking the integral on both sides, and using the substitution $u = -\ln|x|$, we obtain

$$e^{-3\ln|x|}y(x) = \int e^{-3\ln|x|}\left(-\frac{1}{x}\right)dx = \int e^{3u}du = \frac{1}{3}e^{3u} + C = \frac{e^{-3\ln|x|}}{3} + C$$

So, we have

$$y(x) = \frac{1}{3} + Ce^{3\ln|x|} = \frac{1}{3} + C|x|^3$$

Notice that the solutions before and after zero are independent since they are separated by zero. The solutions are therefore given by

$$y(x) = \begin{cases} 1/3 + C_1x^3, & x > 0 \\ 1/3 + C_2x^3, & x < 0 \end{cases}$$

b) The integrating factor is given by

$$e^{F(x)} = e^{\int 2/xdx} = 2\ln x = x^2$$

We multiply both sides of the equation with this and get

$$\frac{d}{dx}(x^2y) = 1.$$

We integrate both sides with respect to x and get

$$x^2y = x + C \implies y = \frac{1}{x} + \frac{C}{x^2}.$$

8. This is the same as solving the initial value problem

$$y' = \frac{2x}{1+y^2}, \quad y(2) = 3$$

which can be rewritten and integrated as

$$\int (1+y^2)dy = \int 2xdx.$$

This gives

$$y + \frac{y^3}{3} = x^2 + C_0$$

after rewriting

$$y^3 + 3y - 3x^2 = C$$

We use our initial data and have

$$C = 3^3 + 3 \cdot 3 - 3 \cdot 2^2 = 24,$$

so the sought after equation is therefore

$$y^3 + 3y - 3x^2 = 24.$$