0.1 What is a differential equation?

A **differential equation** is an equation describing unknown functions through their derivatives. For example

$$\frac{d}{dx}y = 2x$$

is a differential equation where the unknown function y is of the form $y = x^2 + C$ for some C. A slightly more sophisticated example is the equation

$$y'(x) = y(x)$$

which can be solved by the function e^x. The land of differential equations is vast and most of it hard to traverse. We will in this course take a scenic route through the least rough parts of the theory and have a slight look at how modern technology will come to our aid outside of this path.

The first step will be to tell you that we will be looking at only **ordinary differential equations**, ODE for short. This means that all derivatives will be ordinary, i.e.

$$\frac{d}{dx}y(x),\ \frac{d^2}{dx^2}y(x),\ \ldots,\ \frac{d^n}{dx^n}y(x)$$

and not partial, i.e.

$$\frac{\partial}{\partial x}u(x,t),\ \frac{\partial}{\partial t}u(x,t),$$
$$\frac{\partial^2}{\partial x^2}u(x,t),\ \frac{\partial^2}{\partial t^2}u(x,t),\ \frac{\partial^2}{\partial x \partial t}u(x,t),\ \frac{\partial^2}{\partial t \partial x}u(x,t)$$

Equivalently, the unknown functions in this course will only be dependent on one variable. If the variable is time, we will sometimes be using the dot notation, $\dot{y}$, to denote the derivative of y.

Before moving we introduce two extra descriptive words which we will be using. The **order** of a ODE denotes the largest degree of differentiation occurring in the equation. For example $\dot{y} = ay$ which gives us exponential growth for $a > 0$ and exponential decay for $a < 0$, is a first order differential equation. The ODE $mx'' = -kx + mg$ is a second order differential equation, describing a vibrating string.

The notion of **linear** ODE's tells us that the unknown function is given by linear expressions in the equation. That is, $\dot{y} = ay$ is linear, but $\dot{y} = y^2$ is not. In it's most general form a linear ODE looks like

$$y^{(n)} = a_1(t)y^{(n-1)} + a_2(t)x^{(n-2)} + \cdots + a_{n-1}(t)y' + a_n(t)y + g(t)$$

where $y^{(i)}$ is the ith derivative of y and $a_i(t)$ are functions over t.

Solutions

For a function $y(t)$ to be a **solution** of a differential equation, it has to satisfy the equation for every t in some interval I. This interval may be all of $\mathbb{R}$. For time-dependent systems it might also be natural to restrict to the interval of non-negative numbers $[0, \infty)$.

Examples

Differential equations has a broad impact on modern science. They appear frequently in mathematical models attempting to describe real-life phenomenon.

Example 0.1.1 Newton's second law of motion

Newton's second law of motion states that:

> When a body is acted upon by a force, the time rate of change of its momentum equals the force.

Written as a differential equation, the same law states

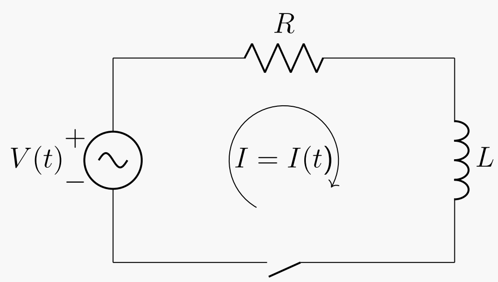
$$m\frac{d}{dt}v = F$$

where m is the mass of the object, v is the speed and F is the acting force. In this equation the unknown function may be the speed of the object v.

More involved differential equations may be found when working on electrical circuits.

Example 0.1.2

Let us look at the RL-circuit



which has a resistance R, an inductance L, and a generator that supplies a voltage

$V(t)$ when the switch is closed. The current $I = I(t)$ in the circuit satisfies the linear first-order ODE

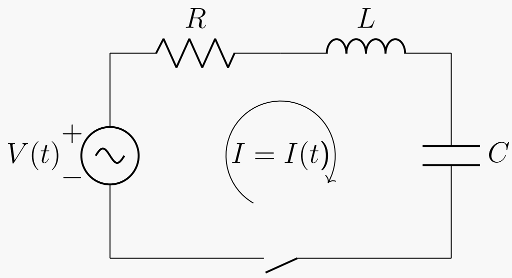
$$L\frac{dI}{dt} + RI = V(t)$$

The general solution for I is given by

$$I(t) = e^{-(R/L)t}\left[c + \frac{1}{L}\int V(t)e^{(R/L)t}dt\right]$$

Example 0.1.3

In the RLC-circuit



The charge $Q = Q(t)$ in the capacitor satisfies the second-order linear non-homogeneous ODE

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t)$$

Chapter 1

First order equations

1.1 First order linear ODE's

We start our journey with one of the simplest kind of differential equations.

Definition 1.1.1

A first-order linear ODE is on the form

$$\frac{dy}{dt} + f(t)y = g(t)$$

where $f(t)$ and $g(t)$ are known functions. Or, written with the dot-notation

$$\dot{y} + f(t)y = g(t)$$

These kind of equations have a straight-forward solutions formula, using what we call the **integrating factor**:

$$e^{F(t)}, \text{ for } F(t) = \int f(t)dt$$

Theorem 1.1.1 Solving first-order linear ODE's

When solving

$$\frac{dy}{dt} + f(t)y = g(t)$$

for f and g continuous on a interval I, you

1. Calculate
$$e^{F(t)} = e^{\int f(t)dt}$$
2. Multiply $e^{F(t)}$ with the equation. The equation now looks like
$$e^{F(t)}\dot{y}(t) + e^{F(t)}f(t)y(t) = e^{F(t)}g(t),$$
3. The left hand side can now be recognized as the product derivative
$$\frac{d}{dt}\left(e^{F(t)}y(t)\right),$$
so the equation is now
$$\frac{d}{dt}\left(e^{F(t)}y(t)\right) = e^{F(t)}g(t).$$
4. Integrating both sides gives
$$e^{F(t)}y(t) = \int e^{F(t)}g(t)\ dt + C,$$
5. Solving with respect to $y(t)$ now gives us the general solution
$$\begin{aligned} y(t) &= e^{-F(t)}\left(\int e^{F(t)}g(t)\ dt + C\right) \\ &= e^{-F(t)}\int e^{F(t)}g(t)\ dt + e^{-F(t)}C \end{aligned} \tag{1.1}$$

It is best to learn the method, rather than to try and remember formula 1.1.

Example 1.1.1

Let us illustrate the method by an example. We solve the following ODE
$$\dot{y} - \frac{1}{4}y = \frac{1}{8}$$
which is on the form $\dot{y} + f(t)y = g(t)$ with $f(t) = -\frac{1}{4}$ and $g(t) = \frac{1}{8}$. Let us start by calculation the integrating factor:
$$e^{F(t)} = e^{\int -1/4dt} = e^{-1/4t}.$$

Observe that we set $C = 0$ in the undetermined integral $\int f(t)dt$. We only need an antiderivative of f in the exponent, so we can do this in general. We multiply the equation with our factor and get

$$\begin{aligned} e^{-1/4t}\dot{y} - e^{-1/4t}\frac{1}{4}y &= e^{-1/4t}\frac{1}{8} \\ \frac{d}{dt}\left(e^{-1/4t}y\right) &= \frac{1}{8}e^{-1/4t} \\ e^{-1/4t}y &= \int \frac{1}{8}e^{-1/4t}dt \\ e^{-1/4t}y &= -\frac{1}{2}e^{-1/4t} + C \\ y(t) &= -\frac{1}{2} + Ce^{1/4t} \end{aligned}$$

Observe that we have a solution for every value of C.

Example 1.1.2

We solve the equation

$$y' = ay$$

where $a > 0$.

This can be written as

$$y' - ay = 0,$$

and we multiply this by e^{-at}, giving

$$e^{-at}y'(t) - e^{-at}ay(t) = 0,$$

and thus

$$\frac{d}{dt}\left(e^{-at}y(t)\right) = 0.$$

After integration, we have

$$e^{-at}y(t) = C,$$

or rather

$$y(t) = Ce^{at},$$

Example 1.1.3

We now consider the problem

$$y'(t) - \cos(t)y(t) = e^{\sin t}.$$

We multiply by

$$e^{-\int \cos t \; dt} = e^{-\sin t},$$

giving

$$\frac{d}{dt}\left(e^{-\sin t}y(t)\right) = 1.$$

After integration, we get

$$e^{-\sin t}y(t) = t + C,$$

os the answer is

$$y(t) = te^{\sin t} + Ce^{\sin t}.$$

Often we know what the unknown function evaluates to for a particular point in time. For example if we are looking for the current $I = I(t)$ in an electrical curcuit, we might assume that at $t = 0$ there is no current at all, i.e. $I(0) = 0$. If we have such constraints on the solution we have an **initial value problem**, or IVP for short.

Example 1.1.4

Find the general solution of

$$\dot{y} + \frac{1}{t} \cdot y = t^2 + 1$$

for $t > 0$. Find also the solution to the initial value problem we get when also assuming $y(1) = 1$.

Integrating factor:

$$e^{F(t)} = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$$

Multiplying t and the equation gives us:

$$\begin{aligned} t\dot{y} + y &= t^3 + t \\ \frac{d}{dt}(ty(t)) &= t^3 + t \\ ty(t) &= \int (t^3 + t) dt \\ y(t) &= \frac{1}{4}t^3 + \frac{1}{2}t + \frac{1}{t}C \end{aligned}$$

Which is the general solution. Now, if $y(1) = 1$ we see that

$$1 = \frac{1}{4} \cdot 1^3 + \frac{1}{2} \cdot 1 + C \implies C = \frac{1}{4}$$

The particular solution for the initial value problem is thus

$$y(t) = \frac{1}{4}t^3 + \frac{1}{2}t + \frac{1}{4t} = \frac{1}{4}(t^3 + 2t + 1)$$

1.2 Existence and uniqueness

In the last example we saw that when adding an extra constraint on the ODE we got a unique solution. This holds in general and we put it in a nice little box to commemorate it.

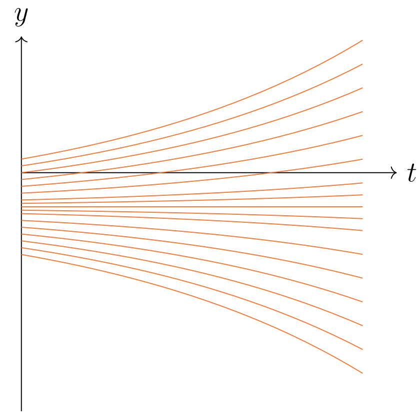
Theorem 1.2.1

Assume that f and g are continuous functions on the open interval I. Let t_0 be any number in I and assume that y_0 is any number in $\mathbb{R}$. Then there exists exactly one solution of the differential equation

$$\frac{dy}{dt} + f(t)y = g(t), \quad t \in I$$

such that $y(t_0) = y_0$.

This result tells us that two solutions of a first-order, linear ODE never intersects. So, if we plot every solution, we obtain a collection of curves that "sits beside eachother". Here we give such a plot for a collection of solutions to the ODE of example 1.1.1:



Solution curves of $\dot{y} - \frac{1}{4}y = \frac{1}{8}$

This is an example of a **direction field**, which may be useful to get a geometric intuition on how the solutions of a ODE behaves.

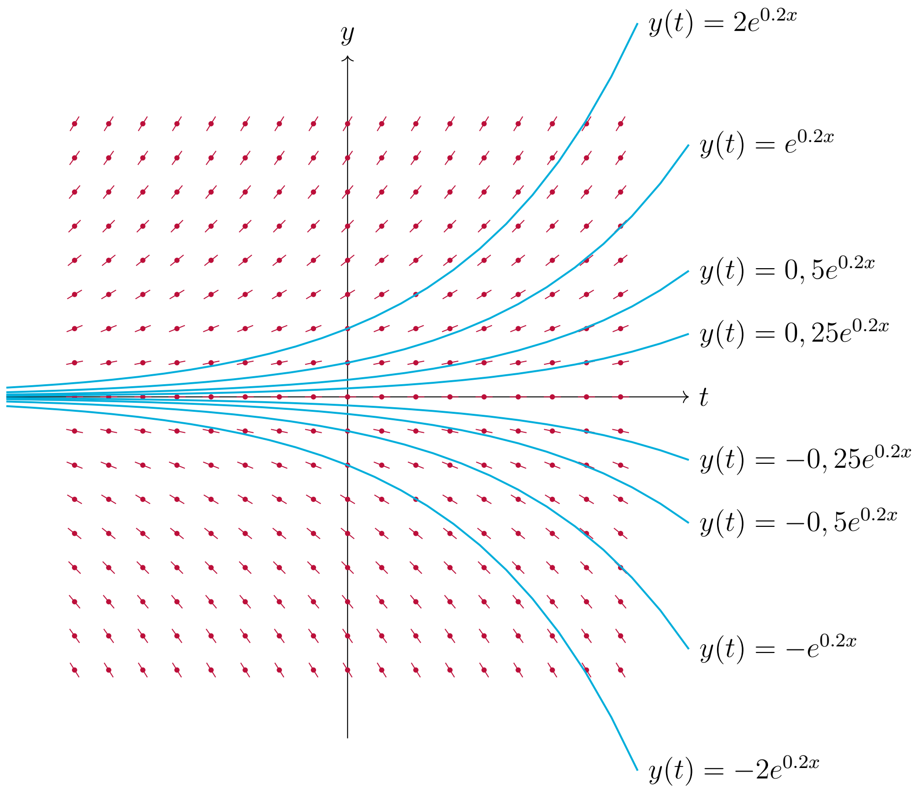
1.3 Direction field

Let us illustrate direction fields through another example. Consider the ODE

$$\dot{y} = ay$$

Let $y = y(t)$ be a solution to this equation. If (t_0, y_0) is a point on the graph of the solution, then we can know from the differential equation that the slope of the solution curve in (t_0, y_0) is given by $\dot{y_0} = ay_0$. If we draw a short line segment in the given point with this slope, we know that the solution cruve has this segment as a tangent. Now, if we draw a such line segments for a grid of points, we can get an idea of how the solutions behave.

Since each line segment is a tangent of a solution, we can use them to sketch possible solution curves. For each point, we know there will only correspond one possible solution.



Direction field of $\dot{y} = 0.2y$ with different solution curves sketched.

Example 1.3.1

Let us sketch the direction field of

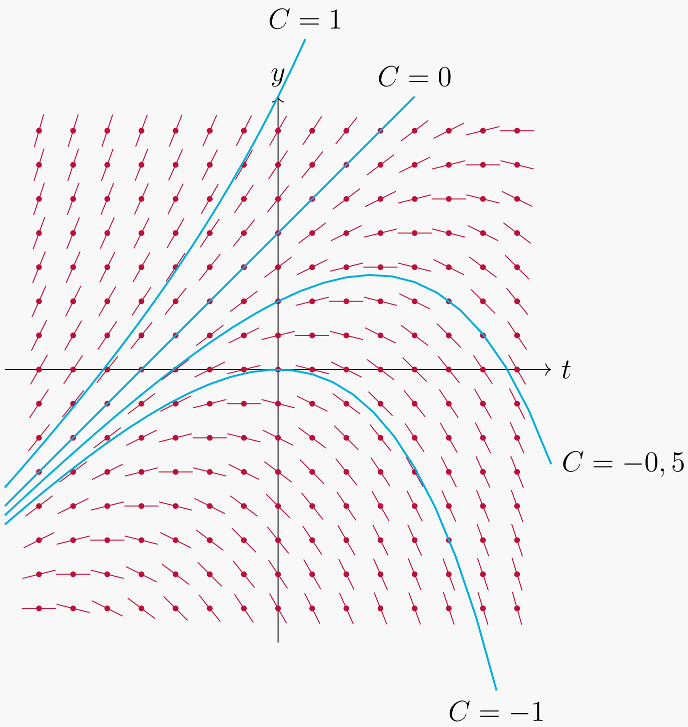
$$\dot{y} = y - t$$

and some solution curves of it.

The general solution of the ODE is

$$y(t) = Ce^t + t + 1$$

and the directional plot with some solution curves is



Equilibrium points

Can we determine the behaviour of the ODE

$$\dot{x}(t) = f(x(t)),$$

without solving the problem explicitly? How does the solution depend on the initial data?

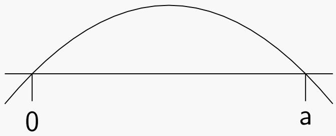
Definition 1.3.1 Equilibrium Point

If $f(t_0) = 0$, t_0 is an **equilibrium point**. These are points at which $\dot{x}(t_0) = 0$, so $x(t)$ stays at x_0 for all t.

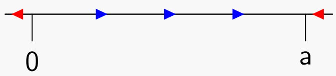
Example 1.3.2

Consider $f(x) = -x(x-a)$, for some $a \in \mathbb{R}$. Then $f(x) = 0$ for $x = 0$ and $x = a$. Thus both of these are equilibrium points. Assume $a > 0$.

What about if we are outside of these equilibrium points? We wish to examine the behaviour in these situations.



Plot of f from Example 1.3.2.

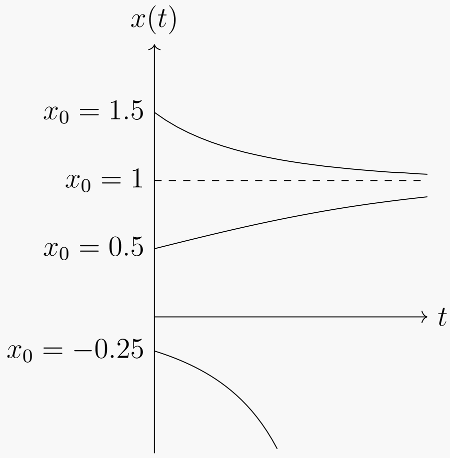


Plot of increasing and decreasing regions
for the solution given some initial data, from Example 1.3.2.

Upon plotting f, we see that there are 3 regions, in which there 2 different behaviours are exhibited. For a choice of $x_0 \in \mathbb{R}$ we have for t near 0 that

- $x_0 < 0$: Then $\dot{x}(t) \approx f(x_0) < 0$, so we are moving to the left,
- $0 < x_0 < a$: Then $\dot{x}(t) \approx f(x_0) > 0$, so we are moving to the right,
- $a < x_0$: Then $\dot{x}(t) \approx f(x_0) < 0$, so we are moving to the left,

These regions are plotted here



We name the equilibrium points based on the behaviour of the solutions near these points:

- the point $x_0 = 0$ is an **unstable** equilibrium point. If the solution starts near here, it will move away over time,
- the point $x_0 = a$ is an **stable** equilibrium point. If the solution starts near here, it will move towards this point over time.

1.4 Separable ODE

Definition 1.4.1
An ODE is **separable** if it can be written on the form

$$q(y(t))\frac{d}{dt}(y(t)) = p(t)$$

with known functions q and p.

The word separable comes from the ability to separate the parts depending on t and the parts depending on y on each side of the equality sign.

Example 1.4.1
The ODE

$$\dot{y} = -\sin(t)y + \sin(t)$$

is separable, since we can rewrite it as

$$\frac{\dot{y}}{1-y} = \sin(t)$$

If $y(t)$ is a solution of this, we get

$$\frac{y'(t)}{1-y(t)} = \sin(t)$$

Integrating with respect to t on both sides gives

$$\int \frac{y'(t)}{1-t}dt = \int \sin(t)dt$$

where the right hand side is recognized as $-\cos(t) + D_1$ for some constant C. What about the left hand side? Well, we can try to substitute $y(t)$ with y and get $y'(t)dt = dy$, so we have

$$\int \frac{1}{1-y} dy = -\ln|1-y| + D_2.$$

Hence, we have

$$\ln|1-y| = \cos(t) + D$$

where we have collected the constants into D, which gives us

$$1 - y = \pm e^D e^{\cos(t)}.$$

That is

$$y(t) = Ce^{\cos(t)} + 1$$

where we have put the pluss/minus sign and e^D into a new unknown C.

The above example also gave us the general solution strategy for solving separable ODE's.

Remark 1.4.1 Solution strategy

Given a separable ODE

$$q(y(t))\frac{d}{dt}(y(t)) = p(t)$$

1. Integrate with respect to t on both sides,

$$\int q(y(t))\frac{d}{dt}(y(t))\,dt = \int p(t)dt$$

2. Substitute $y = y(t)$ in the left integral, which gives $y'(t)dt = dy$,

$$\int q(y)dy = \int p(t)dt$$

3. Do the integration, and solve for y.