Problems for the second day

1. Find the general solution of the equations

a) $y'' - y' - 2y = 0$

b) $y'' + y = 0$

2. What is the general solution of the equation

$$y'' - 3y' + 2y = e^{3t}$$

3. The equation

$$y'' + 2y' + 2y = -2e^{-t}\sin(t),$$

has a particular solution on the form

$$y_p(t) = Ate^{-t}\cos(t) + Bte^{-t}\sin(t)$$

Show that this is in fact a solution if $A = 1$ and $B = 0$
(plug the expression $y_p(t) = te^{-t}\cos(t)$ into the equation).

4. Find a particular solution to each of the equations

a) $y'' - y' - 2y = te^t$

b) $y'' + y = \cos(t)$

5. Solve the initial value problems

a) $y'' - y' - 2y = 0, \quad y(0) = 0$ and $y'(0) = 1$

b) $y'' + y = 0, \quad y\left(\frac{\pi}{2}\right) = 1$ and $y'\left(\frac{\pi}{2}\right) = 0$

6) Solve the initial value problem

$$y'' - 4y' + 13y = 0, \quad y(0) = 1 \text{ and } y'(0) = 5.$$

Rewrite the solution to the form

$$y(t) = Ae^{at}\cos(bt - \phi)$$

Solutions for the second day

1.a The characteristic polynomial is

$$r^2 - r - 2$$

which has the roots

$$r = \frac{1 \pm \sqrt{1 - 4 \cdot (-2)}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

so the general solution is

$$y_h(t) = c_1 e^{2t} + c_2 e^{-t}$$

1.b The characteristic polynomial is

$$r^2 + r = 0$$

which has the roots

$$r = \frac{0 \pm \sqrt{0 - 4}}{2} = \pm i$$

so the general solution is given by

$$y_h(t) = e^{0t}[c_1 \cos(t) + c_2 \sin(t)] = c_1 \cos(t) + c_2 \sin(t)$$

2. The characteristic polynomial is

$$r^2 - 3r + 2$$

which has roots

$$r = \frac{3 \pm \sqrt{9-8}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$$

so the general homogeneous solution is

$$y_h(t) = c_1 e^{2t} + c_2 e^t$$

Now, since we observe that e^{3t} do not solve the homogeneous equation, we try to solve the non-homogeneous equation with a particular solution on the form

$$y_p(t) = Ae^{3t}$$

We put this into the equation and get

$$\begin{aligned} y_p''(t) - 3y_p'(t) + 2y_p(t) &= 9Ae^{3t} - 9Ae^{3t} + 2Ae^{3t} \\ &= 2Ae^{3t} \end{aligned}$$

which tells us that y_p is a particular solution if $A = \frac{1}{2}$. Thus, the general solution is

$$y(t) = y_h(t) + y_p(t) = c_1 e^{2t} + c_2 e^t + \frac{1}{2} e^{3t}$$

3. We start by finding the derivatives

$$\begin{aligned} y_p(t) =& te^{-t} \cos(t) \\ y_p'(t) =& e^{-t} \cos(t) - te^{-t} \cos(t) - te^{-t} \sin(t) \\ y_p''(t) =& -e^{-t} \cos(t) - e^{-t} \sin(t) - e^{-t} \cos(t) + te^{-t} \cos(t) + te^{-t} \sin(t) \\ & - e^{-t} \sin(t) + te^{-t} \sin(t) - te^{-t} \cos(t) \\ =& -2e^{-t} \cos(t) - 2e^{-t} \sin(t) + 2te^{-t} \sin(t), \end{aligned}$$

and therefore

$$\begin{aligned} y_p''(t) + 2y_p'(t) + 2y_p(t) =& -2e^{-t} - 2e^{-t} \sin(t) + 2te^{-t} \sin(t) \\ & + 2e^{-t} \cos(t) - 2te^{=t} \cos(t) - 2te^{-t} \sin(t) \\ & + 2te^{-t} \cos(t) \\ =& -2e^{-t} \sin(t) \end{aligned}$$

which shows that y_p is a solution of the equation.

4.a We know from 1.a) that the general solution of the homogeneous case is given by

$$y_h(t) = c_1e^{2t} + c_2e^{-t}$$

and since te^t can't be found as a solution of this, we will try with a first degree polynomial times e^t as our particular solution,

$$\begin{aligned} y_p(t) =& (At + B)e^t \\ y_p'(t) =& Ate^t + Ae^t + Be^t \\ =& Ate^t + (A + B)e^t \\ y_p''(t) =& Ate^t + Ae^t + (A + B)e^t \\ =& Ate^t + (2A + B)e^t \end{aligned}$$

We put this into the equation and get

$$\begin{aligned} y''(t) - y'(t) - 2y(t) &= (Ate^t + (2A + B)e^t) - (Ate^t + (A + B)e^t) - 2(Ate^t + Be^t) \\ &= -2Ate^t + (2A + B - A - B - 2B)e^t. \end{aligned}$$

We know that this should be equal $te^t + 0$, so we obtain the equations

$$\begin{aligned} -2A =& 1 \\ 0 =& 2A + B - A - B - 2B \\ =& A - 2B \end{aligned}$$

by comparing the two expressions. The first equation tells us that $A = -\frac{1}{2}$, and after putting this into the second equation we get $B = \frac{A}{2} = -\frac{1}{4}$. The particular solution is

$$y_p(t) = -\frac{1}{2}te^t - \frac{1}{4}e^t$$

We could also evaluate the equation

$$te^t = -2Ate^t + (2A + B - A - B - 2B)e^t$$

for two values of t, say $t = 0$ and $t = 1$ to obtain two equations

$$\begin{aligned} t = 0&: \quad 0 = 2A + B - A - B - 2B \\ t = 1&: \quad e = -2Ae + (2A + B - A - B - 2B)e \end{aligned}$$

and solve these two for A and B.

4.b We know from 1.b) that the homogeneous solution is given by

$$y_h(t) = c_1\cos(t) + c_2\sin(t)$$

and since $\cos(t)$ is a homogeneous solution, we try the particular solution

$$\begin{aligned} y_p(t) &= At\cos(t) + Bt\sin(t) \\ y_p'(t) &= A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t) \\ &= (A + Bt)\cos(t) + (B - At)\sin(t) \\ y_p''(t) &= B\cos(t) - (A + Bt)\sin(t) - A\sin(t) + (B - At)\cos(t) \\ &= (2B - At)\cos(t) - (2A + Bt)\sin(t) \end{aligned}$$

We put this into the equation and get

$$\begin{aligned} y'' + y =& (2B - At)\cos(t) - (2A + Bt)\sin(t) + At\cos(t) + Bt\sin(t) \\ =& 2B\cos(t) - 2A\sin(t) = \cos(t) \end{aligned}$$

which tells us that $A = 0$ and $B = \frac{1}{2}$. Our particular solution is therefore

$$y_p(t) = \frac{1}{2}t\sin(t)$$

5.a We know that the general solution is

$$y_h(t) = c_1e^{2t} + c_2e^{-t}$$

We use the information to find the equations

$$\begin{aligned} c_1 + c_2 &= y_h(0) = 0 \\ 2c_1 - c_2 &= y_h'(0) = 1 \end{aligned}$$

By combining these two equations we get that $2c_1 + c_1 = 1$, that is $c_1 = \frac{1}{3}$. This in turn gives us $c_2 = -\frac{1}{3}$. The solution of the IVP is therefore

$$y(t) = \frac{1}{3}(e^{2t} - e^{-t})$$

5.b We know that the general solution is

$$y_h(t) = c_1\cos(t) + c_2\sin(t)$$

This tells us that $y(\pi/2) = c_2 = 1$. After differentiating we have

$$y'\left(\frac{\pi}{2}\right) = -c_1\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) = -c_1 = 0$$

Thus our solution to the IVP is

$$y(t) = \sin(t)$$

6. The characteristic polynomial is

$$r^2 - 4r + 13$$

which has roots

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \begin{cases} 2 + 3i \\ 2 - 3i \end{cases}$$

so the general solution is

$$y_h(t) = e^{2t}[c_1 \cos(3t) + c_2 \sin(3t)]$$

Using the first condition $y(0) = 1$, we get at once that $c_1 = 1$. After differentiating we have

$$\begin{aligned} y_h'(t) &= 2e^{2t}[\cos(3t) + c_2 \sin(3t)] + e^{2t}[-3\sin(3t) + 3c_2 \cos(3t)] \\ &= e^{2t}[(2 + 3c_2)\cos(3t) + (2c_2 - 3)\sin(t)] \end{aligned}$$

and using the second condition $y'(0) = 5$, we then have $2 + 3c_2 = 5$. Thus $c_2 = 1$, and our solution is

$$y(t) = e^{2t}[\cos(3t) + \sin(3t)]$$

Now, to get this to the form $y(t) = Ae^{2t}\cos(3t - \phi)$, we use the formulas

$$A = \sqrt{1^2 + 1^2} = \sqrt{2}$$

and

$$\phi = \arctan\left(\frac{1}{1}\right) = \frac{1}{4}\pi$$

which gives us

$$y(t) = \sqrt{2}e^{2t}\cos\left(3t - \frac{1}{4}\pi\right)$$