Chapter 1

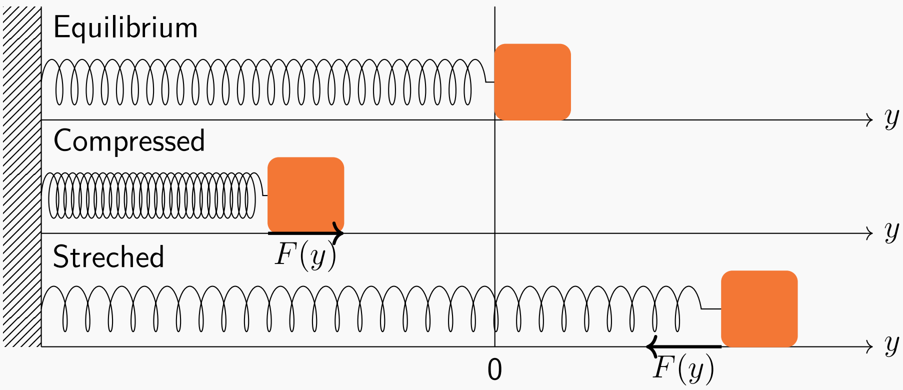
Second order equations

We now consider equations of the form

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t). \tag{1.1}$$

These are **second order linear ordinary differential equations**.

Example 1.0.1



A box connected to the wall by a spring is moving without friction along a surface. Hooke's law tells us that the restoring force in the spring is proportional to the displacement, that is

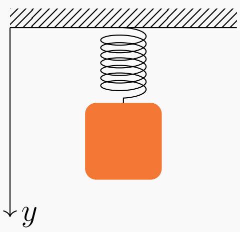
$$F(y) = -ky,$$

where y is the displacement of the box from the equilibrium state of the spring, k is the spring constant, and $F(y)$ is the force acting on the weight from the spring.

If $y(t)$ is the position of the box, then the acceleration of the box is $y''(t)$, and from Newton's second law of motion we have

$$-ky = my'',$$

where m is the mass of the box. This is a second order linear differential equation. Note that as any good physicist we have assumed that we work in a frictionless vacuum[a].



Now, let us assume that the box is connected to the ceiling. Then the gravitational pull will act with a constant force mg downwards. The total force is

$$F(y, y') = -ky - mg,$$

and Newton's second law of motion gives the ODE

$$my'' + ky = mg.$$

[a]https://xkcd.com/669/

In general we may not be able to find a solution to initial value problems of second order ODEs, but we will be working on **second order ODE's with constant coefficients**, which do give us unique solutions:

$$y''(t) + a_1 y'(t) + a_0 y(t) = f(t)$$

We will assume that y is two times differentiable, and defined on all real numbers. In practice, y will only be defined over some interval, but the methods will stay the same.

We will in addition assume that a_0 is nonzero. If one of them is zero, we can easily reduce the problem to a first order equation.

The ODE's are sorted into two categories the *homogeneous*

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

with zero on the right hand side, and the *inhomogeneous*

$$y''(t) + a_1 y'(t) + a_0 y(t) = f(t)$$

where $f(t)$ is a continuous function different from zero.

1.1 Homogeneous Equations

Differentiation is a linear operation, so we obtain at once a result which is often called the **superposition principle**.

Theorem 1.1.1 superposition principle

If $y_1(t)$ and $y_2(t)$ are both solutions to

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

then any scaled sum of them is also a solution, i.e. $c_1 y_1(t) + c_2 y_2(t)$ is a solution for all real numbers c_1 and c_2.

Proof. Let $y_1(t)$ and $y_2(t)$ be two solutions. Take any scaled sum of them $c_1 y_1(t) + c_2 y_2(t)$. Then

$$\begin{aligned} &(c_1 y_1(t) + c_2 y_2(t))'' \\ =& c_1 y_1''(t) + c_2 y_2''(t) \text{ differentiation is linear} \end{aligned}$$

Now use that both $y_1(t)$ and $y_2(t)$ are solutions:

$$\begin{aligned} &(c_1 y_1 + c_2 y_2)'' + a_1 (c_1 y_1 + c_2 y_2)' + a_2 (c_1 y_1 + c_2 y_2) \\ =& c_2 (y_1'' + a_1 y_1' + a_2 y_1) + c_2 (y_2'' + a_1 y_2' + a_2 y_2) \\ =& c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

The function $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is also a solution. □

We have seen that as soon as we have one or two solutions to our equation, we can use the superposition principle to generate an infinite amount of them. Let us now find a way to generate these first two solutions. We try first with an exponential function $y(t) = e^{rx}$. Substituting this into the equation gives

$$\begin{aligned} 0 =& y''(t) + a_1 y'(t) + a_0 y(t) \\ =& r^2 e^{rt} + a_1 r e^{rt} + a_0 e^{rt} \\ =& e^{rt}(r^2 + a_1 r + a_0) \end{aligned}$$

Since e^{rt} is non-zero for all t and r, we see that we need $r^2 + a_1 r + a_0 = 0$. We call this quadratic equation the **characteristic polynomial** to the ODE, and we know that the roots are given by

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

We know that these roots may be on one of three forms: two real roots, one double root or two imaginary roots. Let us look at each of these cases separately.

Two real roots

From the discussion above we see that when we have two different real roots, r_1 and r_2, of the characteristic polynomial, we have the two solutions

$$y_1 = e^{r_1 t} \quad \text{and} \quad y_2 = e^{r_2 t}$$

From the superposition principle we also know that any scaled sum of these are also solutions to the ODE. In fact, we have that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is the general solution of the ODE. We will not show that these are all the possible solutions though. Hopefully you trust me, or find some suitable references to consult.

Example 1.1.1

a) Find the general solution to

$$y'' - y' - 6y = 0$$

The characteristic polynomial of the equation is $r^2 - r - 6$ which have the roots

$$r = \frac{1 \pm \sqrt{1+24}}{2} = \begin{cases} -2 \\ 3 \end{cases}$$

so the general solution is

$$y = c_1 e^{-2t} + c_2 e^{3t}$$

b) Find the general solution to

$$y(t)'' - y(t) = 0.$$

The characteristic polynomial $r^2 - 1$ has the roots $r = 1$ and $r = -1$. The general solution is therefore

$$y(t) = c_1 e^{t} + c_2 e^{-t}.$$

Do notice that in the general solution we have two unknown values c_1 and c_2, so in order to find a solution to a initial value problem of a second order ODE we will need two conditions to get only one possible solution.

Example 1.1.2

a) Solve the initial value problem

$$y''(t) - y(t) = 0$$

with conditions

$$y(0) = 1, \quad y'(0) = 0.$$

We found the general solution

$$y(t) = c_1 e^t + c_2 e^{-t}.$$

and after differentiation

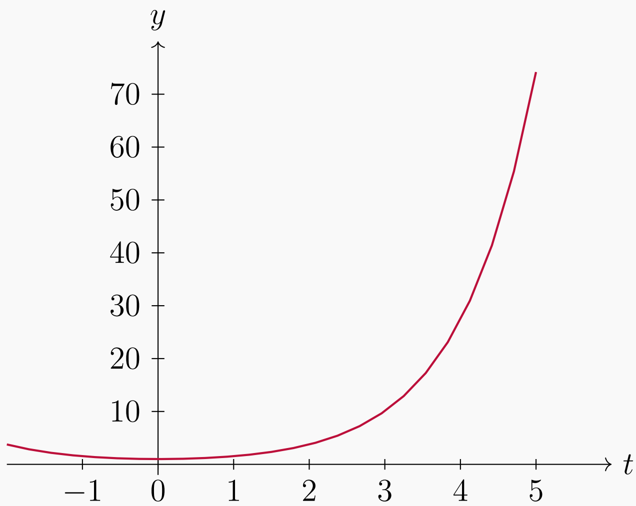
$$y'(t) = c_1 e^t - c_2 e^{-t}.$$

When imposing the conditions on this, we obtain the following two equations

$$\begin{array}{ccccccc} 1 & = & y(0) & = & c_1 & + & c_2 \\ 1 & = & y'(0) & = & c_1 & - & c_2 \end{array}$$

Solving these gives $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$. The solution is

$$y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}.$$



b) Solve the initial value problem

$$y'' - y' - 6y = 0$$

with conditions

$$y(0) = 2, \quad y'(0) = 1.$$

We found the general solution

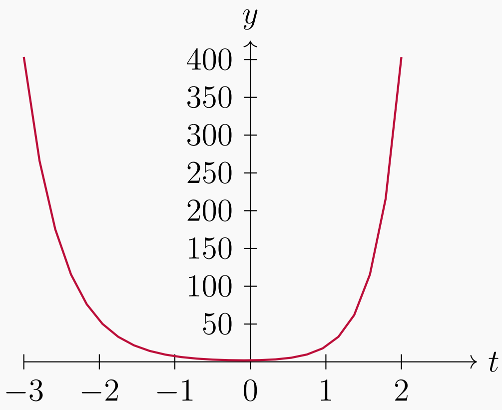
$$y = c_1 e^{-2t} + c_2 e^{3t}$$

and obtain the equations

$$\begin{array}{ccccccc} 2 & = & y(0) & = & c_1 & + & c_2 \\ 1 & = & y'(0) & = & -2c_1 & + & 3c_2 \end{array}$$

Which gives $c_1 = 1$ and $c_2 = 1$, so the solutions is

$$y(t) = e^{-2t} + e^{3t}$$



Double root

If we have only one real root r_1, then we only get one solution from the characteristic polynomial, namely

$$y_1 = e^{r_1 t}.$$

To get a general solution to the problem we need to find another, independent solution as well. We won't dwell to long on what this other solution may be, but instead reveal that it is $y_2 = te^{r_1 t}$. To verify this we observe that since r_1 is the only root of the polynomial, we have

$$r^2 + a_1 r + a_0 r = (r - r_1)^2 = r^2 - 2r_1 r + r_1^2$$

so $a_1 = -2r_1$ and $a_0 = r_1^2$. The ODE can thus be written as

$$y'' - 2r_1 y' + r_1^2 y = 0$$

Before substituting y_2 on the left we calculate the derivatives of y_2,

$$\begin{aligned} y_2' =& e^{r_1 t} + r_1 t e^{r_1 t} \\ y_2'' =& 2r_1 e^{r_1 t} + r_1^2 t e^{r_1 t}. \end{aligned}$$

Now, we get

$$\begin{aligned} y_2'' - 2r_1 y_2' + r_1 y_2 &= (te^{r_1 t})'' - 2r_1 (te^{r_1 t})' + r_1^2 e^{r_1 t} \\ &= 2r_1 e^{r_1 t} + r_1^2 t e^{r_1 t} - 2r_1 (e^{r_1 t} + r_1 t e^{r_1 t}) + r_1^2 e^{r_1 t} \\ &= 0 \end{aligned}$$

We now have two solutions

$$y_1 = e^{r_1 t} \quad \text{and} \quad y_2 = te^{r_1 t}$$

to the ODE, and as with the case of two real roots we tell at once that

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$

is the general solution. Those who can't accept this without a proof is once again asked to consult some suitable reference.

Example 1.1.3

a) Find the general solution to

$$y(t)'' + 2y(t)' + y(t) = 0.$$

The characteristic equation is

$$r^2 + 2r + 1$$

hhich only has one root $r = -1$. Then the general solution is

$$y(t) = c_1 t e^{-t} + c_0 e^{-t}.$$

b) Solve the initial value problem

$$y'' - 4y' + 4y = 0$$

with the conditions

$$y(1) = 0, \quad y'(1) = 2$$

$$r = \frac{4 \pm \sqrt{16 - 16}}{2} = 2$$

The general solution is

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}$$

and the derivative of this is

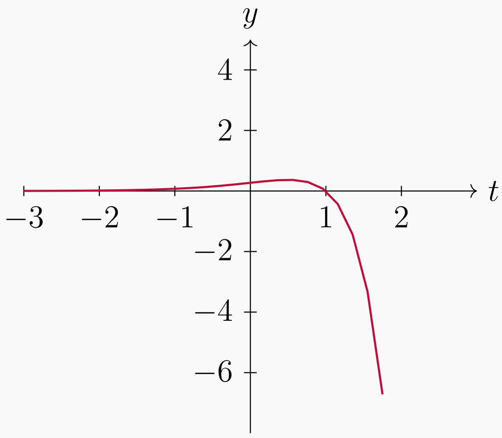
$$y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} = 2c_1 e^{2t} + c_2(1 + 2t)e^{2t}$$

We obtain the following equations

$$\begin{array}{ccccccc} 0 & = & y(1) & = & c_1 e^2 & + & c_2 e^2 \\ 2 & = & y'(1) & = & 2c_1 e^2 & + & 3c_2 e^2 \end{array}$$

which solves to $c_1 = 2e^{-2}$ and $c_2 = -2e^{-2}$. The solution of the initial value problem is therefore

$$y(t) = 2e^{2t-2} - 2te^{2t-2}$$



c) Solve the initial value problem

$$y'' + y' + 0.25y = 0$$

with the conditions

$$y(0) = 3, \quad y'(0) = -3.5$$

$$r = \frac{-1 \pm \sqrt{1 - 4 \cdot 0.25}}{2} = -\frac{1}{2} = -0.5$$

The general solution is

$$y(t) = c_1 e^{-0.5t} + c_2 t e^{-0.5t}$$

and the derivative of this is

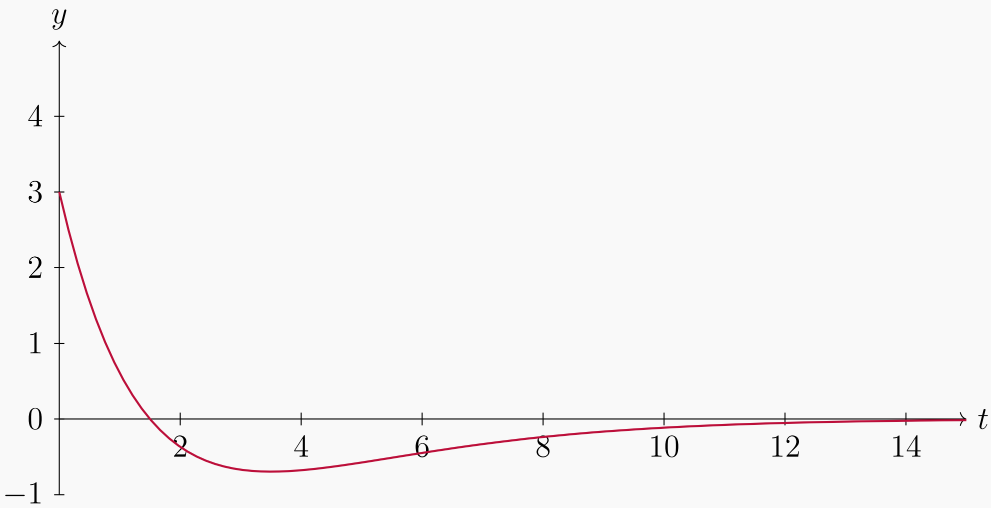
$$y'(t) = c_2 e^{-0.5t} - 0.5(c_1 + c_2 t)e^{-0.5t}.$$

Substituting $t = 0$ and using the initial condition we get

$$y(0) = c_1 = 3, \quad y'(0) = c_2 - 0.5c_1 = -3.5 \quad \implies \quad c_2 = -2$$

The solution of the initial value problem is therefore

$$y(t) = 3e^{-0.5t} - 2te^{-0.5t}$$



Two complex roots

Now, the last case is two complex roots. We have implicitly assumed throughout that our differential equation is real-valued and specifically that a_1 and a_0 are real numbers. Then we know that the roots are complex conjugate of each other,

$$r_1 = a + ib, \quad r_2 = a - ib$$

We know that

$$e^{(a+ib)t} = e^{at}e^{ibt}, \quad \text{and} \quad e^{(a-ib)t} = e^{at}e^{-ibt}$$

are solutions to our ODE, but as we are mostly interested in real-valued solutions we want to dismiss the complex ones. The first step to do this is to recall **Euler's formula** which tells us that

$$e^{ib} = \cos(b) + i\sin(b).$$

After a bit of algebraic manipulation we see that

$$\frac{e^{ibt} + e^{-ibt}}{2} = \cos(bt), \quad \text{and} \quad \frac{e^{ibt} - e^{-ibt}}{2i} = \sin(bt).$$

From this, we obtain the following two real-valued solutions

$$\begin{aligned} y_1(t) &= \frac{1}{2}e^{(a+ib)t} + \frac{1}{2}e^{(a-ib)t} \\ &= e^{at}\frac{e^{ib} + e^{-ibt}}{2} \\ &= e^{at}\cos(bt) \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= \frac{1}{2i}e^{(a+ib)t} - \frac{1}{2i}e^{(a-ib)t} \\ &= e^{at}\frac{e^{ib} - e^{-ibt}}{2i} \\ &= e^{at}\sin(bt) \end{aligned}$$

The general real-valued solution of the ODE having complex roots $a \pm ib$ to the characteristic polynomial is on the form

$$y(t) = e^{at}[c_1\cos(bt) + c_2\sin(bt)].$$

Once again, if you want a proof that these are all the real-valued solution you should consult some suitable reference.

Example 1.1.4

a) Find the general real-valued solution of

$$y(t)'' + y(t) = 0.$$

The characteristic polynomial $r^2 + 1$ have the complex roots $r = i$ and $\overline{r} = -i$.

That is $a = 0$ and $b = 1$, so the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t.$$

b) Find the general solution of

$$y'' + 2y' + 4y = 0$$

The characteristic polynomial $r^2 + 2r + 4$ have the complex roots $r = -1 \pm i\sqrt{3}$. That is $a = -1$ and $b = \sqrt{3}$, so the general solution is

$$y(t) = e^{-t}[c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)]$$

c) Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0$$

with the conditions

$$y(0) = 0, \quad y'(0) = -3$$

$$r = \frac{-0.4 \pm \sqrt{0.16 - 36.16}}{2} = \begin{cases} -0.2 + 3i \\ -0.2 - 3i \end{cases}$$

That is, $a = -0.2$ and $b = 3$, so the general solution is

$$y(t) = e^{-0.2t}[c_1 \cos(3t) + c_2 \sin(3t)]$$

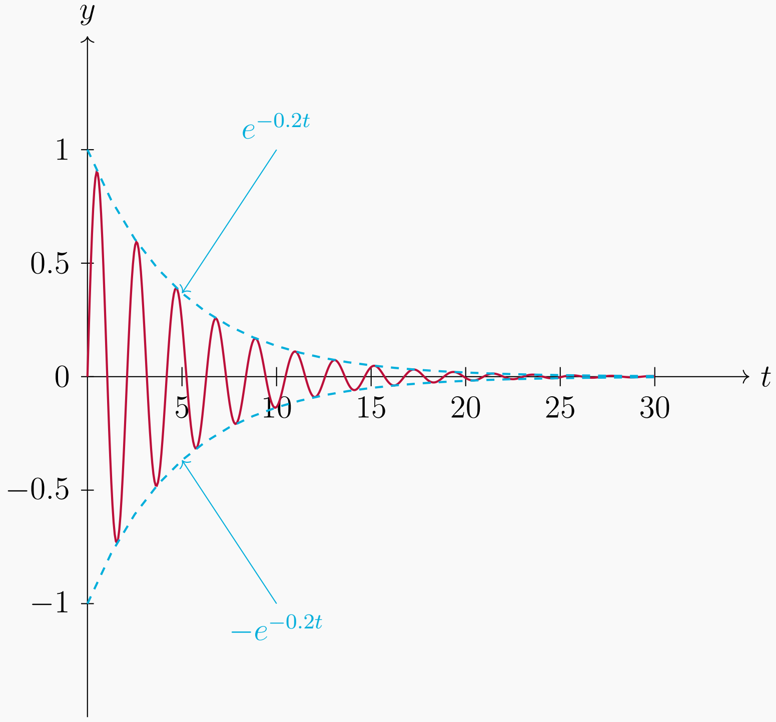
At once we observe that the condition $y(0) = 0$ forces $c_1 = 0$ since $\cos(0) = 1$. To determine c_2 we know differentiate the resulting expression

$$y'(t) = c_2[-0.2e^{-0.2t} \sin(3t) + 2e^{-0.2t} \cos(3t)].$$

By substituting in $t = 0$ and using the second condition we get $y'(0) = 3c_2 = 3$ sp $c_2 = 1$. The particular solution is

$$e^{-0.2t} \sin(3t).$$

$$y(t) = 3e^{-0.5t} - 2te^{-0.5t}$$



c) Solve the initial value problem

$$y'' - 0.2y' + 16.01y = 0$$

with the conditions

$$y(0) = 1, \quad y'(0) = 4.1$$

$$r = \frac{0.2 \pm \sqrt{0.04 - 4 \cdot 16.01}}{2} = \begin{cases} 0.1 + 4i \\ 0.1 - 4i \end{cases}$$

That is, $a = 0.1$ and $b = 4$, so the general solution is

$$y(t) = e^{0.1t}[c_1 \cos(4t) + c_2 \sin(4t)]$$

From the condition $y(0) = 1$ we see that $c_1 = 0$, so we have

$$y(t) = e^{0.1t}[\cos(4t) + c_2 \sin(4t)]$$

and in order to obtain c_2 we differentiate:

$$y'(t) = e^{0.1t}[(4c_2 + 0.1)\cos(4t) + (0.1c_2 - 4)\sin(t)].$$

Substituting $t = 0$ we get

$$y'(0) = 4c_2 + 0.1 = 4.1$$

so $c_2 = 1$ and the particular solution is

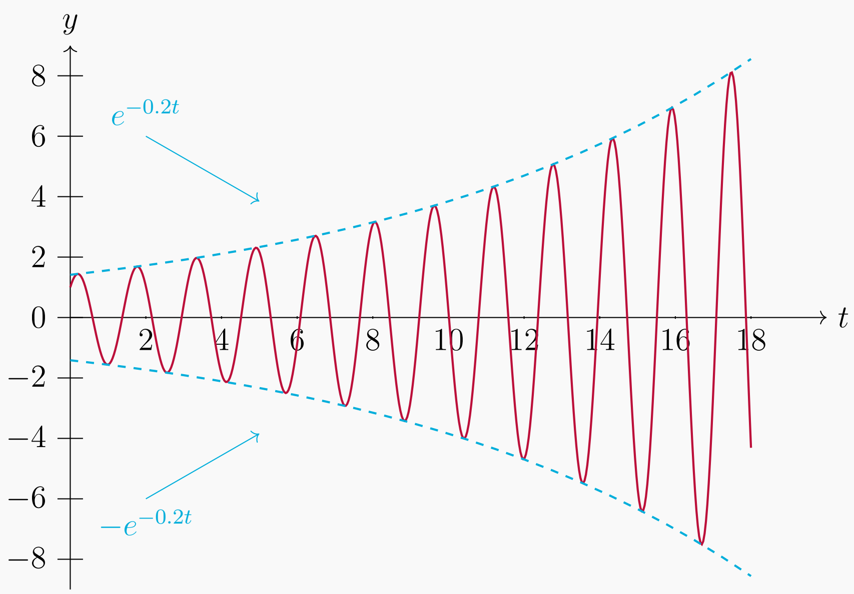
$$y = e^{0.1t}[\cos(4t) + \sin(4t)]$$

Now, since the cosine and sine has the same period given by 4, we can rewrite the expression in the bracket as a cosine with some amplitude and a phase shift, i.e.

$$\cos(4t) + \sin(4t) = A\cos(4t - \phi)$$

for some amplitude A and some phase ϕ. As will be remarked in the summary the amplitude A is given by $\sqrt{1^2 + 1^2} = \sqrt{2}$, and the phase ϕ is given by $\arctan(1/1) = \pi/4$, hence our solution can also be written as

$$y(t) = \sqrt{2}e^{0.1t}\cos(4t - \pi/4)$$



Summary

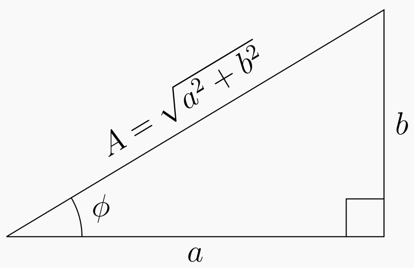
Theorem 1.1.2

The general solution of
$$y'' + a_1 y' + a_0 = 0$$
depends on the roots of the characteristic polynomial
$$r^2 + a_1 r + a_0.$$
The general solution is
$$y(t) = \begin{cases} c_1 e^{r_1 t} + c_2 e^{r_2 t} & \text{if there are two real roots } r_1 \text{ and } r_2. \\ c_1 e^{r_1 t} + c_2 t e^{r_1 t} & \text{if there are is a double root } r_1. \\ e^{at}[c_1 \cos(bt) + c_2 \sin(bt)] & \text{if there are two imaginary roots } a \pm ib. \end{cases}$$

Remark 1.1.1

If we get a particular solution to a second order ODE consisting of some exponential and a function on the form $a\cos(ct) + b\sin(ct)$ we will often like to rewrite this part to get a single cosine-function $A\cos(ct - \phi)$.

We do this by first remembering that if we have a right triangle with catheti (or more commonly; legs), of length a and b



then the hypotenuse has length $\sqrt{a^2 + b^2} = A$ by the Pythagorean Theorem, and the angle between the hypotenuse and the leg of length a has angle ϕ given by $\phi = \arctan \frac{b}{a}$. From the definition of cosine and sine, we have
$$a = A\cos(\phi) \quad \text{and} \quad b = A\sin(\phi).$$
Now, using the identity $\cos(u \pm v) = \cos(u)\cos(v) \mp \sin(u)\sin(v)$ we observe that
$$a\cos(ct) + b\sin(ct) = A\cos(\phi)\cos(ct) + A\sin(\phi)\sin(ct) = A\cos(ct - \phi)$$

However neither a nor b is necessarily positive, but the formulas will hold true up to possibly a difference of π in the angle ϕ, i.e. the angle ϕ lie in the same quadrant as the point (a, b). To summarize

$$a\cos(ct) + b\sin(ct) = A\cos(ct - \phi)$$

where $A = \sqrt{a^2 + b^2}$ and $\phi = \arctan\frac{b}{a}$ $(+\pi)$ lie in the same quadrant as (a, b).

1.2 Inhomogeneous equations

Let us now look at the inhomogeneous case, that is equations on the form

$$y''(t) + a_1 y'(t) + a_0 y(t) = f(t) \tag{1.2}$$

where $f(t)$ is a continuous function, not everywhere equal to zero. We will write $y_p(t)$ for solutions to (1.2) and call them particual solutions.

Assume that we have found a particular solution $y_p(t)$. If $y(t)$ is some other solution to (1.2), then $y(t) - y_p(t)$ is a homogeneous solution, that is, a solution to the corresponding homogeneous equation,

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0.$$

Theorem 1.2.1

Every solution to the inhomogeneous equation is on the form

$$y(t) = y_p(t) + y_h(t)$$

where $y_p(t)$ is a particular solution and $y_h(t)$ is a solution to the corresponding homogeneous equation.

In order to find all the solutions we therefore get the following strategy: find one particular solution and then add the general solution of the homogeneous case.

Finding a particular solution

There is an analytic way to find a particular solution which we will give shortly. However, there is a much easier method that works for a surprising amount of cases which we will focus on.

Remark 1.2.1 Analytic method / Method of variation of parameters

Let $y_1(t)$ and $y_2(t)$ be two independent solutions to the homogeneous equation. Then the particular solution can be found as

$$\begin{aligned} y_p(t) = y_2(t) \int \frac{y_1(t) f(t)}{y_1(t) y_2'(t) - y_2(t) y_1'(t)}\, dt \\ - y_1(t) \int \frac{y_2(t) f(t)}{y_1(t) y_2'(t) - y_2(t) y_1'(t)}\, dt. \end{aligned}$$

Now, let us move on to the method we love and adore: **The method of undetermined coefficients**.

The idea: Look for a solution that looks like the right hand side function $f(t)$ of (1.2) with general coefficients. Determine these coefficients by substituting this into the problem. If the right hand side is of the same form as the solution to the homogeneous problem, multiply by t as in the case with a single root.

This method relies on you seeing examples and know what to look for in each case. It really is a method relying on intuition.

Example 1.2.1

Let $f(t) = K \neq 0$ be a constant and consider the inhomogeneous ODE

$$y''(t) + a_1 y'(t) + a_0 y(t) = K$$

with $a_0 \neq 0$ not zero. We try with a particular solution $y_p(t) = c$ which is also a constant. Substituting into the equation

$$K = y_p''(t) + a_1 y_p'(t) + a_0 y_p(t) = a_0 c.$$

Since $a_0 \neq 0$, we get the particular solution $y_p(t) = K/a_0$.

Example 1.2.2

We consider the equation

$$y'' + 2y' + 2y = -2e^{-t} \sin t. \tag{1.3}$$

The characteristic polynomial is

$$r^2 + 2r + 2$$

and so the roots are given by completing the square,

$$(r+1)^2+1=0.$$

So we have two roots $r_\pm = -1 \pm i$. The basis for the homogeneous problem solution space is thus

$$y_1(t) = e^{-t}\cos t, \text{ and } y_2(t) = e^{-t}\sin t.$$

If the right hand side was not already part of the solution for the homogeneous problem, we would try for the particular solution

$$y_p(t) = Ae^{-t}\cos t + Be^{-t}\sin t$$

and determine A and B. Instead we will try

$$y_p(t) = Ate^{-t}\cos t + Bte^{-t}\sin t.$$

Substituting this into (1.4), we find that

$$2e^{-t}(B\cos(t) - A\sin(t)) = -2e^{-t}\sin t,$$

and so we find $B=0$ and $A=1$. Thus the general solution is

$$\begin{aligned} y(t) &= y_p(t) + c_1y_1(t) + c_2y_2(t) \\ &= te^{-t}\cos t + c_1e^{-t}\cos t + c_2e^{-t}\sin t, \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

Example 1.2.3

Consider the ODE

$$y''(t) + y(t) = e^t.$$

We try $y_p(t) = ce^t$ with a constant c. Substituting $y_p(t)$ in the equation, tells us that $c = 1/2$.

Example 1.2.4

We consider the inhomogeneous ODE

$$y''(t) - y(t) = \cos t.$$

We try $y_p(t) = a\cos t + b\sin t$ with constants a and b. Note that we have to include both $\cos t$ and $\sin t$ even when $f(t)$ only contains $\cos t$. Substituting $y_p(t)$ in the equation:

$$\begin{aligned}\cos t &= y_p''(t) - y_p(t)\\ &= (a\cos t + b\sin t)'' - (a\cos t + b\sin t)\\ &= -a\cos t - b\sin t - a\cos t - b\sin t\\ &= -2a\cos t - 2b\sin t.\end{aligned}$$

The left and right hand side is only equal for all t if $a = -\frac{1}{2}$ and $b = 0$. The general solution to the inhomogeneous equation is

$$y(t) = c_1te^t + c_2e^{-t} - \frac{1}{2}\cos t.$$

Example 1.2.5

Consider the inhomogeneous ODE

$$y''(t) + 2y'(t) + y(t) = t^2 - 1.$$

The right hand side $f(t)$ is a polynomial. Thus we try with a polynomial of the same degree as $f(t)$: $y_p(t) = at^2 + bt + c$ with constants a, b, c. Note that we include the part bt even though it do not appear in $f(t)$. Substitute $y_p(t)$ in the equation:

$$\begin{aligned}t^2 - 1 &= y_p''(t) + 2y_p(t) + y_p(t)\\ &= 2a + 2(2at + b) + (at^2 + bt + c)\\ &= 2a + 4at + 2b + at^2 + bt + c\\ &= at^2 + (4a + b)t + 2a + 2b + c.\end{aligned}$$

The left and right hand side are only equal if

$$\begin{aligned}1 &= a\\ 0 &= 4a + b\\ -1 &= 2a + 2b + c.\end{aligned}$$

Hence $a = 1$, $b = -4$ and $c = 5$, that is

$$y_p(t) = t^2 - 4t + 5.$$

The general solution of the inhomogeneous ODE is therefore on the form

$$y(t) = c_1 t e^{-t} + c_2 e^{-t} + t^2 - 4t + 5.$$

Example 1.2.6

We consider the equation

$$y'' + 2y' + 2y = -2e^{-t} \sin t. \tag{1.4}$$

The characteristic polynomial is

$$r^2 + 2r + 2$$

and so the roots are given by completing the square,

$$(r+1)^2 + 1 = 0.$$

So we have two roots $r_\pm = -1 \pm i$. The basis for the homogeneous problem solution space is thus

$$y_1(t) = e^{-t} \cos t, \text{ and } y_2(t) = e^{-t} \sin t.$$

If the right hand side was not already part of the solution for the homogeneous problem, we would try for the particular solution

$$y_p(t) = Ae^{-t} \cos t + Be^{-t} \sin t$$

and determine A and B. Instead we will try

$$y_p(t) = Ate^{-t} \cos t + Bte^{-t} \sin t.$$

Substituting this into (1.4), we find that

$$2e^{-t}(B \cos(t) - A \sin(t)) = -2e^{-t} \sin t,$$

and so we find $B = 0$ and $A = 1$. Thus the general solution is

$$\begin{aligned} y(t) &= y_p(t) + c_1 y_1(t) + c_2 y_2(t) \\ &= te^{-t} \cos t + c_1 e^{-t} \cos t + c_2 e^{-t} \sin t, \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

Example 1.2.7

Consider the equation

$$y''(t) + 2y'(t) + y(t) = e^{-t}.$$

We can't have the particular solution $y_p(t) = ce^{-t}$, since it is also a solution to the homogeneous equation:

$$\begin{aligned} y''(t) + 2y'(t) + y(t) &= (ce^{-t})'' + 2(ce^{-t})' + ce^{-t} \\ &= ce^{-t} - 2ce^{-t} + ce^{-t} \\ &= 0. \end{aligned}$$

We tried in the example above to add on a factor t to remedy this, but here this is also a solution of the homogeneous equation. Then we add on another factor t and try $y_p(t) = ct^2e^{-t}$:

$$\begin{aligned} e^{-t} &= y_p''(t) + 2y_p'(t) + y_p(t) \\ &= (ct^2e^{-t})'' + 2(ct^2e^{-t})' + ct^2e^{-t} \\ &= ce^{-t}(t^2 - 4t + 2 + 4t - 2t^2 + t^2) \\ &= 2ce^{-t}. \end{aligned}$$

since $e^{-t} \neq 0$ for all t, we need c to be equal to $1/2$. We are left with $y_p(t) = \frac{1}{2}t^2e^{-t}$. The general solution of the inhomogeneous ODE is then on the form

$$y(t) = c_1e^{-t} + c_2te^{-t} + \frac{1}{2}t^2e^{-t}.$$

Even though this method is to some degree based on intuition, we list a few good guiding rules here.

Remark 1.2.2

1. **Basic rule:** If $f(t)$ in (1.2) is one of the functions in the first column in Table 1.2, choose the trial solution $y_p(t)$ in the second column.

2. **Modification:** If a term of you choice for $y_p(t)$ is a solution of the homogeneous ODE, multiply it with t. If the solution correspond to a double root, multiply with t^2.

3. **Sum rule:** If $f(t)$ is a sum of functions from the first column of Table 1.2, choose $y_p(t)$ as a sum of the corresponding trial solutions.

Forcing function $f(t)$	**Trial solution** $y_p(t)$	**Comment**
Ke^{ct}	ae^{ct}	
$K\cos(\omega t)$ or $K\sin(\omega t)$	$a\cos(\omega t) + b\sin(\omega t)$	
$p(t)$	$q(t)$	p is a polynomial; q is a polynomial of same degree
$p(t)\cos(\omega t)$ or $p(t)\sin(\omega t)$	$q(t)\cos(\omega t) + r(t)\sin(\omega t)$	p is a polynomial; q & r are polynomials of same degree
$Ke^{ct}\cos(\omega t)$ or $Ke^{ct}\sin(\omega t)$	$e^{ct}[a\cos(\omega t) + b\sin(\omega t)]$	
$e^{ct}p(t)\cos(\omega t)$ or $e^{ct}p(t)\sin(\omega t)$	$e^{ct}q(t)\cos(\omega t) + e^{ct}r(t)\cos(\omega t)$	p is a polynomial; q & r are polynomials of same degree

Table 1.1: Method of Undetermined Coefficients