Chapter 1

Numerical solutions of ODEs

More often than not there exists no analytical solution of ODEs. Here the computer comes to our aid through numerical analysis. There are a great deal of resources available[1] to explore numerical solution methods of ODE's, so we are going to settle with looking at one methods. In fact, we will also restrict ourselves to the first order ODE's.

Let us start the short exposition by looking at what conditions need to be met for our ODE's to have a unique solution.

1.1 Existence and uniqueness

A reasonable question to ask is: How do we know for sure that the solution exists and is unique? We have two theorems for this, the first pertaining to just existence, and the second to both uniqueness and existence.

Theorem 1.1.1 Peano's Existence Theorem

Let $D \subset \mathbb{R}^2$ be an open set and $f : D \to \mathbb{R}$ be a continuous function. Then the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

with $(t_0, x_0) \in D$ has a solution on some open interval I containing x_0.

The solution need not be unique.

[1] One resource is for example https://folk.ntnu.no/leifh/teaching/tkt4140/

Definition 1.1.1 Lipschitz Continuity

f is Lipschitz continuous if there exists an $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|,$$

for all $x, y \in I$.

Lipschitz continuity is a restriction on the growth of function. It states the slope of a function can never grow too large.

Theorem 1.1.2 Picard-Lindelöf Existence and Uniqueness Theorem

Suppose f be locally Lipschitz (Lipschitz on some open set containing 0), then there exists a unique solution of the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

on some open set containing 0 (i.e. on a set $[0 - \epsilon, 0 + \epsilon]$ for some $\epsilon > 0$).

1.2 Euler's Method

Now that we have some criterion for the existence and uniqueness of a solution, let us start by exploring Euler's method of approximating such a solution.

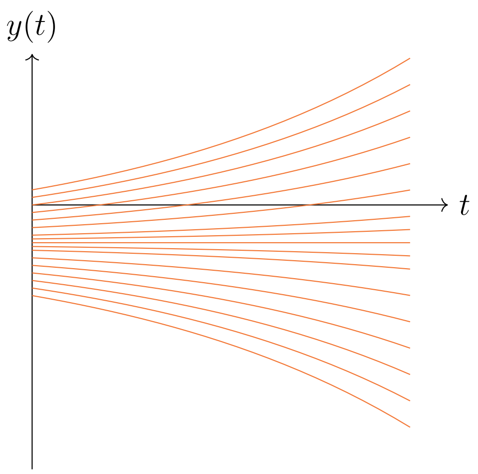


Figure 1.1: Solution curves of $\dot{y} - \frac{1}{4}y = \frac{1}{8}$

We consider an initial value problem

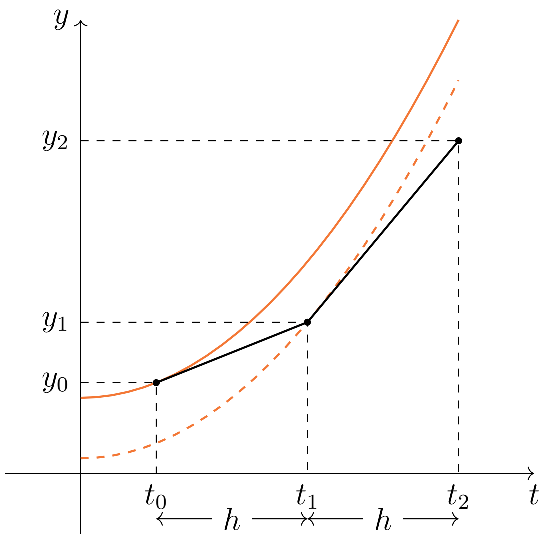
$$\begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases} .$$

We assume f has sufficient assumptions (i.e. those of Picard-Linelöf) such that there is a unique solution to this problem. Through each point (t, y) in the ty-plane there is a possible solution curve of the differential equation.

A possible way to approximate one of these curves is the following: We start in a point (t_0, y_0) and move h steps along the tangent of the solution curve in that point. The point we are currently at is denoted (t_1, y_1). We now go h steps along the tangent of the solution curve going through that point and end up in (t_2, y_2). Once again we move h steps along the tangent of the solution curve in this point and end up in (t_3, y_3). When we iterate the same process further we are left with a list of points

$$(t_0, y_0), (t_1, y_1), (t_2, y_2), (t_3, y_3), \ldots, (t_n, y_n), \ldots$$

Drawing line segments between these points gives an approximation of the solution curve $y(t)$ of the initial value problem, and if we let the step size h move towards zero the approximation will converge to the solution.



Let us summarize the method.

Remark 1.2.1 Euler's method

When using Euler's method with a step size of h to approximate a solution of the initial value problem

$$\begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}.$$

we get a segmented line through the points (t_n, y_n) given iteratively as

$$t_n = t_0 + n \cdot h, \quad y_n = y_{n-1} + f(t_{n-1}, y_n - 1) \cdot h$$

with starting point (t_0, y_0).

Proof. Directly from the definition of the derivative we have,

$$\dot{y}(t_n) = \lim_{h \to 0} \left[\frac{y(t_n + h) - y(t_n)}{h} \right] \approx \frac{y(t_{n+1}) - y(t_n)}{h}.$$

We define the approximate value of the function at time step t_n as

$$y_n \approx y(t_n).$$

From this, using the fact that $\dot{y} = f(t, y)$, we define the method by

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n)h,$$

or after rearranging

$$y_{n+1} = y_n + f(y_n).$$

□

Example 1.2.1

Let us use the Euler's method with step size $h = 1$ to approximate the solution to the initial value problem

$$\begin{cases} \dot{y} = f(t, y) = \frac{1}{8} + \frac{1}{4}y \\ y(0) = \frac{1}{2} \end{cases}$$

We have earlier calculated analytically that the solution to this problem is

$$y(t) = -\frac{1}{2} + e^{\frac{1}{4}t},$$

so we will also plot this and the approximated solution and see how close the approximation is. The initial condition tells us that our starting point (t_0, y_0)

is $(0, \frac{1}{2})$. Let us calculate the first four points.

$$
\begin{aligned}
t_1 &= 1, & y_1 &= y_0 + f(t_0, y_0) \cdot h \\
&& &= \tfrac{1}{2} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{1}{2}\right) \cdot 1 \\
&& &= \tfrac{3}{4}
\end{aligned}
$$

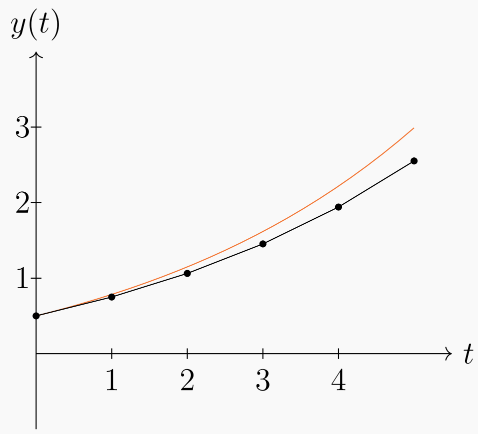
$$
\begin{aligned}
t_2 &= 2, & y_2 &= y_1 + f(t_1, y_1) \cdot h \\
&& &= \tfrac{3}{4} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{3}{4}\right) \cdot 1 \\
&& &= \tfrac{17}{16}
\end{aligned}
$$

$$
\begin{aligned}
t_3 &= 3, & y_3 &= y_2 + f(t_2, y_2) \cdot h \\
&& &= \tfrac{17}{16} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{17}{16}\right) \cdot 1 \\
&& &= \tfrac{93}{64}
\end{aligned}
$$

$$
\begin{aligned}
t_4 &= 4, & y_4 &= y_3 + f(t_3, y_3) \cdot h \\
&& &= \tfrac{93}{64} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{93}{64}\right) \cdot 1 \\
&& &= \tfrac{497}{256}
\end{aligned}
$$

$$
\begin{aligned}
t_5 &= 5, & y_5 &= y_4 + f(t_4, y_4) \cdot h \\
&& &= \tfrac{497}{256} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{497}{256}\right) \cdot 1 \\
&& &= \tfrac{2613}{1024}
\end{aligned}
$$

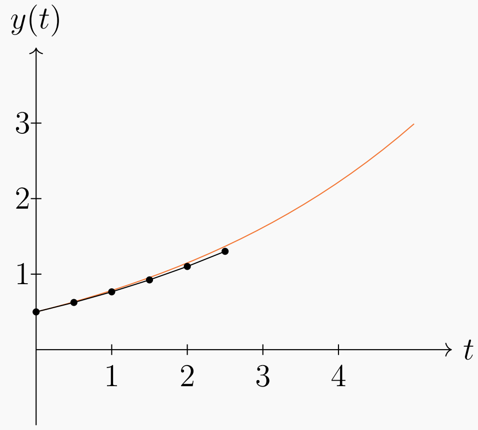
Now, let us plot these points and the exact analytical solution



Let us also try with step size $h = \frac{1}{2}$,

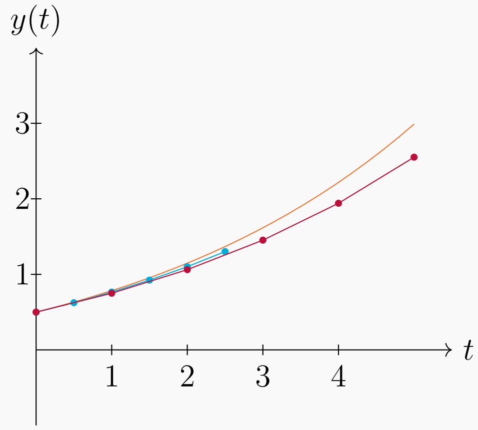
$$
\begin{aligned}
t_1 &= \tfrac{1}{2}, & y_1 &= y_0 + f(t_0, y_0) \cdot h \\
&& &= \tfrac{1}{2} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{1}{2}\right) \cdot \tfrac{1}{2} \\
&& &= \tfrac{5}{8} \\
t_2 &= 1, & y_2 &= y_1 + f(t_1, y_1) \cdot h \\
&& &= \tfrac{5}{8} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{5}{8}\right) \cdot \tfrac{1}{2} \\
&& &= \tfrac{49}{64} \\
t_3 &= 1.5, & y_3 &= y_2 + f(t_2, y_2) \cdot h \\
&& &= \tfrac{49}{64} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{49}{64}\right) \cdot \tfrac{1}{2} \\
&& &= \tfrac{473}{512} \\
t_4 &= 2, & y_4 &= y_3 + f(t_3, y_3) \cdot h \\
&& &= \tfrac{473}{512} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{473}{512}\right) \cdot \tfrac{1}{2} \\
&& &= \tfrac{4513}{4096} \\
t_5 &= 2.5, & y_5 &= y_4 + f(t_4, y_4) \cdot h \\
&& &= \tfrac{4513}{4096} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{4512}{4096}\right) \cdot \tfrac{1}{2} \\
&& &\approx 1.302032
\end{aligned}
$$

Now, let us plot these points and the exact analytical solution.



And now let us plot both approximations, step size $h = 1$ in red and step size

$h = \frac{1}{2}$ in blue.



Remark 1.2.2

Observe on the last step in the example, we only get an approximate value for y_5. This is an example of the float point error one often gets when doing numerical analysis. The computer can only store a certain amount of decimals, so for every step after this we would get increasingly wrong estimations of y, on top of us already doing an approximation of the actual solution.