Problems for fourth day

1. Write the set of equations on matrix form. Find the augmented matrix and solve by Gaussian elimination.

a)

$$\begin{cases} x_1 - 2x_2 - 3x_3 = 0 \\ 2x_2 + x_3 = -8 \\ -x_1 + x_2 + 2x_3 = 3 \end{cases}$$

b)

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 = 13 \\ 3x_1 + 3x_2 + 6x_3 = 15 \end{cases}$$

c)

$$\begin{cases} x - 4y + 28z = -2 \\ -x + y - 7z = -31 \\ x + 2y - 14z = 64 \end{cases}$$

2. Which of these matrices are in row echelon form? Which of them are in reduced row echelon form?

a)

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

3.

a) Solve the set of equations

$$\begin{cases} 2x - y + z = 0 \\ 3x + y - 6z = 0 \\ 4x - 2y + 2z = 0 \end{cases}$$

and

$$\begin{cases} 2x - y + z = 1 \\ 3x + y - 6z = 4 \\ 4x - 2y + 2z = 2 \end{cases}$$

Explain the relation between the solutions.

b) Find values a, b, c such that

$$\begin{cases} 2x - y + z = a \\ 3x + y - 6z = b \\ 4x - 2y + 2z = c \end{cases}$$

don't have any solution.

4. Assume that we have given a set of equation consisting of m equations and n unknowns. Which of the nine cases in the following table is possible?

	$m < n$	$m = n$	$m > n$
0 solutions			
1 solution			
∞ solutions			

5. Determine whether the system of equations given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 5 \\ 1 & -2 & -1 & 1 \\ 0 & -4 & -1 & -1 \end{array}\right]$$

has a solution.

6. Consider a solved Sudoko puzzle as a 9×9-matrix A. Calculate the product

$$A\begin{bmatrix}1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1\end{bmatrix}^T$$

7. Let A and B be matrices, and $\mathbf{v}$ a vector:

$$A=\begin{bmatrix}0 & 1 & 5 \\ 2 & 3 & -1 \\ -8 & 0 & 2\end{bmatrix} \quad B=\begin{bmatrix}1 & 2 & 5 \\ 0 & 0 & 3\end{bmatrix} \quad \mathbf{v}=\begin{bmatrix}7 \\ 2 \\ -4\end{bmatrix}$$

Calculate (or explain why the expression do not make sense):

a) AB d) B^2 g) $BA\mathbf{v}$

b) BA e) $A+B$ h) B^T

c) A^2 f) $(A+I_3)\mathbf{v}$ i) $\mathbf{v}^T\mathbf{v}$

8. Find a 2×2-matrix A such that

$$A\begin{bmatrix}1 \\ 0\end{bmatrix}=\begin{bmatrix}3 \\ 5\end{bmatrix} \quad \text{and} \quad A\begin{bmatrix}1 \\ 1\end{bmatrix}=\begin{bmatrix}-1 \\ 0\end{bmatrix}$$

9. Find, if possible, the inverse matrices of

a)

$$\begin{bmatrix}1 & 7 \\ -1 & 1\end{bmatrix}$$

b)

$$\begin{bmatrix}1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1\end{bmatrix}$$

10.

a) For which values of $a\in\mathbb{R}$ is the matrix

$$A=\begin{bmatrix}1 & 0 & a \\ a & 1 & 0 \\ 0 & a & 1\end{bmatrix}$$

invertible?

b) Verify that

$$A^{-1}=\frac{1}{2}\begin{bmatrix}1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1\end{bmatrix}$$

when $a=1$.

Solutions for fourth day

1. a The system of equations can be written on the form $A\mathbf{x}=\mathbf{b}$, with

$$A=\begin{bmatrix}1 & -2 & -3 \\ 0 & 2 & 1 \\ -1 & 1 & 2\end{bmatrix}, \quad \mathbf{b}=\begin{bmatrix}0 \\ -8 \\ 3\end{bmatrix}$$

Gauss elimination yields

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 1 & -8 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

and we obtain $x + 3$, $x_2 = -5$ and $x_1 = -4$ by backward substitution. The solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$

is unique.

1. b We have

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 3 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 13 \\ 15 \end{bmatrix}$$

By Gaussian elimination we get

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 13 \\ 3 & 3 & 6 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 13 \\ 1 & 0 & 2 & 2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 3 & 0 & 9 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right]$$

we can choose x_3 as we want, let's say $x_3 = t$, and then we get $x_1 = 2 - 2t$ and $x_2 = 3$. The number of solutions are infinite, and they are on the form

$$\mathbf{x} = \begin{bmatrix} 2 - 2t \\ 3 \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

for $t \in \mathbb{R}$.

1. c The system written as an augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -4 & 28 & -2 \\ -1 & 1 & -7 & -31 \\ 1 & 2 & -14 & 64 \end{array}\right]$$

By Gaussian elimination we get

$$\left[\begin{array}{ccc|c} 1 & -4 & 28 & -2 \\ -1 & 1 & -7 & -31 \\ 1 & 2 & -14 & 64 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 42 \\ 0 & 1 & -7 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From row 1 we read directly that $x = 42$. Row 3 means

$$0x + 0y + 0z = 0$$

or $0 = 0$. This do not tell us anything about the values of x, y and z. We have a free variable $z = t$, where t is a real number (we could also have chosen y to be the free variable). With $z = t$ we see in row 2 that $y = 11 + 7t$. The number of solutions are infinite, and they are on the form

$$\mathbf{x} = \begin{bmatrix} 42 \\ 11 + 7t \\ t \end{bmatrix} = \begin{bmatrix} 42 \\ 11 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

for $t \in \mathbb{R}$.

2. Matrix (a) and (c) are in row echelon form; (a) is in row reduced echelon form.

3. a We put the homogeneous system into an augmented matrix and eliminate:

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 3 & 1 & -6 & 0 \\ 4 & -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & \frac{5}{2} & -\frac{15}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The third variable is free, we put it as $x_3 = t$. Then we see in the second row that $x_2 = 3t$, and in the first row that $x_1 = t$. We have the solution

$$x = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} t, \quad \forall t \in \mathbb{R}.$$

In the next system, the inhomogeneous one, we get a nearly identical augmented matrix, except in the fourth column:

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & 1 & -6 & 4 \\ 4 & -2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & \frac{5}{2} & -\frac{15}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Again, we obtain a free variable $x_3 = t$. The second row then tells us that $x_2 = 1 + 3t$, and the first row that $x_1 = 1 + t$. In vector notation, the solution

$$x = \begin{bmatrix} 1+t \\ 1+3t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} t, \quad \forall t \in \mathbb{R}.$$

3. b We must find a vector $\mathbf{b} = [b_1, b_2, b_3]^T$ such that the system $A\vec{x} = \mathbf{b}$ can't be solved, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -6 \\ 4 & -2 & 2 \end{bmatrix}$$

This happens if and only if we have an impossible equation in the reduced system. There are a lot of possibilities, but we notice that in the third row of our matrix, which represent $4x - 2y + 2z$, is two times the first row. So let us try a vector $\mathbf{b}$ where b_3 is not equal to $2b_1$? For example $\mathbf{b} = [0, 0, 1]^T$,

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 3 & 1 & -6 & 0 \\ 4 & -2 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & \frac{5}{2} & -\frac{15}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Here we have a contradiction in the third row, for no values of x, y, z can solve $0x + 0y + 0z = 1$. The vector $\mathbf{b} = [0, 0, 1]^T$ is therefore one of many possible answers to the problem.

4.

Let 1 denote possible and 0 impossible:

	$m < n$	$m = n$	$m > n$
0 solutions	1	1	1
1 solution	0	1	1
∞ solutions	1	1	1

Explanation: In each case we either find an example, or explain why no examples exists.

No solutions: Regardless the amount of equations and unknowns, we can always construct an example with an equation that says $0 = 1$.
Infinitely many solutions: We need to verify that we can always construct an example with a free variable. In the case $m < n$ we can take the 1×2 system $x + y = 0$, where y is free. For $m = n$ we can add the equation $0 = 0$ so we have two equations with two unknowns, and y still free. For $m > n$ we can add the equation $0 = 0$ again.
Exactly one solution: For $m = n$ we can take the system $x = 1$, which has a unique solution. If you don't accept this as a proper system you can add on $y = 1$. To obtain an example for $m > n$ we add on the equation $0 = 0$. Now, we have to think, "What happens if we have more unknowns than equations?". We can construct the augmented matrix of the system, this is wider than it is high, since we have more unknowns than equation. Thus we can't get a pivot element in each column. Each column without a pivot element gives a free variable. Hence the system has either no solutions or infinitely many.

5.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 5 \\ 1 & -2 & -1 & 1 \\ 0 & -4 & -1 & -1 \end{array}\right] \xrightarrow{III=III+II} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 5 \\ 2 & 0 & 2 & 6 \\ 0 & -4 & -1 & -1 \end{array}\right]$$

If we add row two to row three, we see that we have $x + y = 0$ and $2x + 2y = 6$, which can't both be true. Thus the system has no solution.

6. In a Sudoku puzzle every row has to contain exactly one copy of each number between 1 and 9, so when you multiply with $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$, you will in each term get a sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

in each entry, thus

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 45 \\ 45 \\ 45 \\ 45 \\ 45 \\ 45 \\ 45 \\ 45 \\ 45 \end{bmatrix}$$

Our solved Sudoku puzzle could for example be

9	4	8	5	7	6	3	2	1
7	5	1	9	3	2	4	8	6
3	6	2	4	8	1	5	7	9
8	1	7	6	9	5	2	3	4
5	2	9	7	2	4	1	6	8
4	2	6	3	1	8	9	5	7
6	9	5	2	4	7	8	1	3
2	8	4	1	6	3	7	9	5
1	7	3	8	5	9	6	4	2

7.

a) Mismatching sizes.

b) $\begin{bmatrix} -36 & 7 & 13 \\ -24 & 0 & 6 \end{bmatrix}$

c) $\begin{bmatrix} -38 & 3 & 9 \\ 14 & 11 & 5 \\ -16 & -8 & -36 \end{bmatrix}$

d) Mismatching sizes.

e) Mismatching sizes.

f) $\begin{bmatrix} -11 \\ 26 \\ -68 \end{bmatrix}$

g) $\begin{bmatrix} -290 \\ -192 \end{bmatrix}$

h) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 5 & 3 \end{bmatrix}$

i) 69

8. $A\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the first column of A, while $A\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the sum of the columns in A, i.e.

$$A = \begin{bmatrix} 3 & x \\ 5 & y \end{bmatrix}$$

where $3 + x = -1$ og $5 + y = 0$. Thus

$$A = \begin{bmatrix} 3 & -4 \\ 5 & -5 \end{bmatrix}$$

9. a We put the identity matrix on the side of the matrix, and reduce it to reduced row echelon form:

$$\begin{bmatrix} 1 & 7 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 1 & 0 \\ 0 & 8 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{8} & \frac{-7}{8} \\ 0 & 1 & \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

so the inverse is

$$\begin{bmatrix} \frac{1}{8} & \frac{-7}{8} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

9. b The method from a) gives the inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

10. a The matrix A is invertible whenever it has maximum rank (which is 3). By Gaussian elimination we find

$$\begin{bmatrix}1 & 0 & a\\ a & 1 & 0\\ 0 & a & 1\end{bmatrix} \sim \begin{bmatrix}1 & 0 & a\\ 0 & 1 & a^2\\ 0 & a & 1\end{bmatrix}$$
$$\sim \begin{bmatrix}1 & 0 & a\\ 0 & 1 & -a^2\\ 0 & 0 & 1+a^3\end{bmatrix}$$

From the echelon form we see that A has maximum rank if and only if $1 + a^3 \neq 0$, thus whenever $a \neq -1$. Thus the matrix is invertible if and only if $a \neq -1$.

10. b Multiply the two matrices and check that you get the identity matrix.