Problems for fifth day

1. Given the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \mathbf{y} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 2 \\ 2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -1 \\ 0 \end{bmatrix},$$

determine which of the following sets are linearly independent

a) $\{\mathbf{x}, \mathbf{v}\}$

b) $\{\mathbf{x}, \mathbf{w}\}$

c) $\{\mathbf{y}, \mathbf{v}\}$

d) $\{\mathbf{y}, \mathbf{w}\}$

e) $\{\mathbf{x}, \mathbf{y}, \mathbf{v}\}$

2. Given the vectors

$$\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix} 0 \\ 2 \\ 2 \end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

a) Are $\{\mathbf{x}, \mathbf{y}, \mathbf{v}\}$ a linearly independent set?

b) Is the vector

$$\mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

in the subspace span$\{\mathbf{x}, \mathbf{y}, \mathbf{v}\}$?

c) Solve, if possible, the vector equation

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{v} = \mathbf{w}$$

3. Find the Column space, Row Space and Null space of the following matrices

a)

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

d)

$$\begin{bmatrix} 0 & 1 & 5 \\ 2 & 3 & -1 \\ -8 & 0 & 2 \end{bmatrix}$$

e)

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix}$$

4. Determine the determinant of the following matrices

a)

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$$

b)

$$\begin{bmatrix} 2 & 2 & 4 \\ 0 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 10 & 1 & 2 & 3 \end{bmatrix}$$

d)

$$\begin{bmatrix} 1 & 0 & 4 & 2 & 2 \\ 2 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

e)

$$\begin{bmatrix} 5 & 9 \\ 2 & 3 \end{bmatrix}$$

5. Is the following matrices invertible?

a)

$$\begin{bmatrix} 1 & 7 \\ -1 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

6. Let A be the matrix

$$\begin{bmatrix} a & b & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & y & z \end{bmatrix}$$

a) Find an expression of $\det(A)$ with a, b, c, x, y, z as variables.

b) Determine for which values

$$a, b, c, x, y, z$$

the matrix A is invertible.

Solutions for fifth day

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$ is linearly independent if the sum

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_t\mathbf{v}_t$$

equals the zero vector only when $a_1, a_2, \ldots, a_t$ are all zero. This is equal to

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_t \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{bmatrix} = \mathbf{0}$$

only having the trivial solution, i.e.

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{bmatrix} = \mathbf{0}.$$

Thus, we can check linearly independence of a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$, by reducing the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_t \end{bmatrix}$$

and check if all columns have a pivot-element. We also remember that for quadratic matrices, $n \times n$, every column has a pivot element if and only if the determinant is nonzero. Hence, if we have n vectors in $\mathbb{R}^n$, we can simply check the determinant of

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_t \end{bmatrix}.$$

1.

a)

$$\begin{bmatrix}\mathbf{x} & \mathbf{v}\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \implies \text{linearly independent.}$$

or

$$\begin{vmatrix}\mathbf{x} & \mathbf{v}\end{vmatrix} = \begin{vmatrix}1 & 2\\0 & 2\end{vmatrix} = 1\cdot 2 - 0\cdot 2 = 2 \neq 0 \implies \text{linearly independent.}$$

b)

$$\begin{bmatrix}\mathbf{x} & \mathbf{w}\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \implies \text{linearly dependent.}$$

or

$$\begin{vmatrix}\mathbf{x} & \mathbf{w}\end{vmatrix} = \begin{vmatrix}1 & -1\\0 & 0\end{vmatrix} = 1\cdot 0 - 0\cdot(-1) = 0 \implies \text{linearly dependent.}$$

c)

$$\begin{bmatrix}\mathbf{y} & \mathbf{v}\end{bmatrix} = \begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} \implies \text{linearly dependent.}$$

or

$$\begin{vmatrix}\mathbf{y} & \mathbf{v}\end{vmatrix} = \begin{vmatrix}1 & 2\\1 & 2\end{vmatrix} = 1\cdot 2 - 1\cdot 2 = 0 \implies \text{linearly dependent.}$$

d) $\{\mathbf{y}, \mathbf{w}\}$

$$\begin{bmatrix}\mathbf{y} & \mathbf{w}\end{bmatrix} = \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \implies \text{linearly independent.}$$

or

$$\begin{vmatrix}\mathbf{y} & \mathbf{w}\end{vmatrix} = \begin{vmatrix}1 & -1\\1 & 0\end{vmatrix} = 1\cdot 0 - 1\cdot(-1) = -1 \neq 0 \implies \text{linearly independent.}$$

e)

$$\begin{bmatrix}\mathbf{x} & \mathbf{y} & \mathbf{v}\end{bmatrix} = \begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\end{bmatrix}$$

We have more columns than rows, and since the number of rows is the maximal amount of pivot elements, we can't have a pivot element in each column. Thus, the set is linearly independent.

Theorem 0.0.1

If we have vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$, and $m < n$, then the vectors are linearly dependent.

We can also state this as

Remark 0.0.1

If U is a subspace of V, and V has dimension m, then

$$\dim(U) \leq m.$$

2.

a)

$$\begin{aligned}
\begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{v} \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & 2 & 3 \end{vmatrix} \\
&= 1 \cdot \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} \\
&= 1 \cdot (2 \cdot 3 - 2 \cdot 2) - 0 \cdot (2 \cdot 3 - 0 \cdot 2) + 1 \cdot (2 \cdot 2 - 0 \cdot 2) \\
&= 2 + 0 + 4 = 6 \neq 0
\end{aligned}$$

The determinant is nonzero, so the vectors are linearly independent.

b) We see that $U = \text{span}(\mathbf{x}, \mathbf{y}, \mathbf{v})$ is given by all possible sums of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{v}$, also since these vectors are linearly independent we know they form a basis for the subspace U. Thus the dimension of U is 3, and since U lie as a subspace of $\mathbb{R}^3$, we have $\mathbb{R}^3 = U$. Now,

$$\mathbf{w} \in \mathbb{R}^3 = U$$

so $\mathbf{w}$ is in u.

Alternatively, we could observe that for $\mathbf{w}$ to lie in U, there must exist values a, b, c such that

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{v} = \mathbf{w}$$

or equivalently, that the equation

$$\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{v} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{w}$$

has a solution. Now, we use that the determinant of the matrix is non-zero, so there exists a unique solution of this equation. Hence $\mathbf{w}$ lie in U.

c) We construct the augmented matrix and reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 2 & 2 & 2 & 5 \\ 0 & 2 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & -3 \\ 0 & 2 & 3 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & \frac{-3}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

so the solution is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 3 \end{bmatrix}$$

and

$$\mathbf{w} = \mathbf{x} - \frac{3}{2}\mathbf{y} + 3\mathbf{v}$$

3.

a) This matrix is already in reduced row echelon form, so we can read out at once that the basis of the column space is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

i.e. $\text{Col}(A) = \mathbb{R}^2$. The basis of the row space is

$$\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

To find the basis of the null space, we choose x_2 and x_3 as our free variables and get the general solution of the homogeneous equation as

$$\begin{array}{llllll} x_1 + & 5x_2 & & & = 0 \\ & x_2 & & & = t \\ & & x_3 & & = s \\ & & & x_4 & = 0 \end{array} \implies \begin{array}{l} x_1 = -5t \\ x_2 = t \\ x_3 = s \\ x_4 = 0 \end{array}$$

or on vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and a basis for the null space is

$$\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

b) Basis Col(A):

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Basis Row(A):

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Basis Null(A):

$$\emptyset$$

c)

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis Col(A):

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

Basis Row(A):

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

Basis Null(A):

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

d)

$$\begin{bmatrix} 0 & 1 & 5 \\ 2 & 3 & -1 \\ -8 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 5 \\ -8 & 0 & 2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 5 \\ 0 & 12 & -2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & -62 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Basis Col(A):

$$\begin{bmatrix} 0 \\ 2 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

Alternatively, since the matrix has max rank, we could have chosen the basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis Row(A):

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis Null(A):

$$\emptyset$$

e)

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Basis Col(A):

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Basis Row(A):

$$\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Basis Null(A):

$$\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

4.

a)

$$\begin{vmatrix}1 & 0\\0 & 2\end{vmatrix} = 1 \cdot 2$$

b)

$$\begin{vmatrix}2 & 2 & 4\\0 & 1 & 3\\0 & 0 & 3\end{vmatrix} = 2 \cdot 1 \cdot 3$$

c)

$$\begin{vmatrix}0 & 0 & 0 & 0\\1 & 2 & 3 & 4\\5 & 6 & 7 & 8\\10 & 1 & 2 & 3\end{vmatrix} = 0 \cdot \begin{vmatrix}2 & 3 & 4\\6 & 7 & 8\\1 & 2 & 3\end{vmatrix} - 0 \cdot \begin{vmatrix}1 & 3 & 4\\5 & 7 & 8\\10 & 2 & 3\end{vmatrix} + 0 \cdot \begin{vmatrix}1 & 2 & 4\\5 & 6 & 8\\10 & 1 & 3\end{vmatrix} - 0 \cdot \begin{vmatrix}1 & 2 & 3\\5 & 6 & 7\\10 & 1 & 2\end{vmatrix}$$
$$= 0 - 0 + 0 - 0 = 0$$

d)

$$\begin{vmatrix}1 & 0 & 4 & 2 & 2\\2 & 0 & 2 & 0 & 3\\0 & 1 & 3 & 1 & 0\\0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 1 & 0\end{vmatrix}$$

We start by choosing row 5, and by the checker pattern

$$\begin{matrix}+ & - & + & - & +\\- & + & - & + & -\\+ & - & + & - & +\\- & + & - & + & -\\+ & - & + & - & +\end{matrix}$$

we get first that

$$\begin{vmatrix} 1 & 0 & 4 & 2 & 2 \\ 2 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 0 & 4 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

We now choose the second column, and using checker pattern

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

we get

$$\begin{vmatrix} 1 & 0 & 4 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

We choose the third column, and using the checker pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

we get

$$\begin{aligned} \begin{vmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} &= 0 \cdot \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} \\ &= -1(-1) + 1(-6) = -5 \end{aligned}$$

And in conclusion

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 4 & 2 & 2 \\ 2 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} &= -1 \cdot \begin{vmatrix} 1 & 0 & 4 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} \\ &= (-1) \cdot (-1) \cdot \begin{vmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} \\ &= (-1)(-1)(-5) = -5 \end{aligned}$$

e)

$$\begin{vmatrix} 5 & 9 \\ 2 & 3 \end{vmatrix} = 15 - 18 = -3$$

5.

a)

$$\begin{vmatrix} 1 & 7 \\ -1 & 1 \end{vmatrix} = 1 + 7 = 8$$

The determinant is nonzero, so the matrix is invertible.

b)

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 = 1$$

The determinant is nonzero, so the matrix is invertible.

6. a We follow the second column:

$$\begin{vmatrix} a & b & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & y & z \end{vmatrix} = -b \begin{vmatrix} c & 0 & 0 \\ 0 & 0 & x \\ 0 & y & z \end{vmatrix} + 0 \cdot \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & x \\ 0 & y & z \end{vmatrix} - 0 \begin{vmatrix} a & 0 & 0 \\ c & 0 & 0 \\ 0 & y & z \end{vmatrix} + 0 \begin{vmatrix} a & 0 & 0 \\ c & 0 & 0 \\ 0 & 0 & x \end{vmatrix}$$

$$= -b \cdot \left(c \cdot \begin{vmatrix} 0 & x \\ y & z \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & x \\ 0 & z \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 0 \\ 0 & y \end{vmatrix} \right)$$

$$- b \cdot c \cdot (-xy) = bcxy$$

6. b The matrix is invertible as long as b, c, x and y is nonzero. a and z can be whatever.