

Linear independence and subspaces

1.1 Linear independence

Linear independence in $\mathbb{R}^3$

If we have two vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ in $\mathbb{R}^3$, both non-zero, there are two possibilities

- I. They are parallel, i.e. $\mathbf{v}_1 = a\mathbf{v}_2$ for some scalar $a \neq 0$. then

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2\} = \text{span}\{\mathbf{v}_1\} = \text{span}\{\mathbf{v}_2\}.$$

and we say that the two vectors are **linearly dependent**.

- II. They are *not* parallel. Then

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$$

is a plane in $\mathbb{R}^3$ containing the origin. We say that the two vectors are **linearly independent**.

In case II, we introduce a third non-zero vector $\mathbf{v}_3$. Once again we have two possibilities:

- II(a) $\mathbf{v}_3$ lies in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$, i.e. $\text{span}\{\mathbf{v}_1, \mathbf{v}_2\}$. Equivalently $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$. All vectors lie in the same plane, and we say that the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.
- II(b) $\mathbf{v}_3$ do *not* lie in the plane spanned by $\mathbf{v}_1$ and $\mathbf{v}_2$. Then $\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is the whole of $\mathbb{R}^3$, and the three vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent.

Linear independence in $\mathbb{R}^n$

Definition 1.1.1

The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are **linearly independent** if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

only have the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. Otherwise they are **linearly dependent**.

A set of vectors are linearly independent only if the only way of combining them into the zero vector is to choose all scales to be zero.

Example 1.1.1

Consider

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

in $\mathbb{R}^3$. The equation

$$\mathbf{v}_1x_1 + \mathbf{v}_2x_2 = \mathbf{0}$$

that is

$$\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} x_2 = \begin{bmatrix} 3x_1 \\ -2x_1 \\ 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives $3x_1 = 0$ og $4x_2 = 0$, so $x_1 = x_2 = 0$. The equation only has the trivial solution, so $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear independent.

Example 1.1.2

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \begin{bmatrix} 4 \\ 4 \\ 9 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

These vectors satisfy the following equality:

$$\mathbf{u} + 5 \cdot \mathbf{v} - 9 \cdot \mathbf{w} = \mathbf{0}$$

and is therefore linear dependent.

Checking for independence

To check if a collection of vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ are linearly independent, we solve the equation

$$\mathbf{v}_1 \cdot x_1 + \mathbf{v}_2 \cdot x_2 + \dots + \mathbf{v}_n \cdot x_n = \mathbf{0}$$

The left hand side can be recognized as a matrix with columns $\mathbf{v}_i$ multiplied with a vector with elements x_i . The equation is therefore equal to

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This can be solved as usual by reducing the augmented matrix

$$\left[\mathbf{v}_1 \quad \mathbf{v}_2 \quad \dots \quad \mathbf{v}_n \mid \mathbf{0} \right]$$

Example 1.1.3

Are these vectors linear independent?

$$\mathbf{u} = \begin{bmatrix} 3 \\ 9 \\ 3 \\ 3 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 2 \\ 4 \end{bmatrix} \quad \text{and} \quad \mathbf{w} = \begin{bmatrix} 8 \\ 31 \\ 12 \\ 22 \end{bmatrix}$$

The equation is

$$\mathbf{u} \cdot x + \mathbf{v} \cdot y + \mathbf{w} \cdot z = \mathbf{0},$$

Reducing the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 2 & 8 & 0 \\ 9 & 7 & 31 & 0 \\ 3 & 2 & 12 & 0 \\ 3 & 4 & 22 & 0 \end{array} \right] \sim \left[\begin{array}{ccc|c} 3 & 2 & 8 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array} \right]$$

We see that we will obtain a pivot element in each of the columns, and thus no free variables. Then we only have one solution: $x = y = z = 0$. We conclude that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

The essential information in the reduction to establish linear (in)dependence is the existence of free variables. Do note that we do not need to include the right hand side in the reduction. There are only zeros in the right hand side from the beginning, so they will stay zero regardless of the row operation done.

Theorem 1.1.1

Let A be a matrix. The following statements are equivalent:

1. The columns of A are linearly independent.
2. The equation $A\mathbf{x} = \mathbf{0}$ is only solved by the trivial solution $\mathbf{x} = \mathbf{0}$.
3. There are no free variables in the solution of $A\mathbf{x} = \mathbf{0}$.
4. When we reduce A , we get a pivot element in every column.

Remark 1.1.1

Theorem 1.1.1 gives us a good method of checking for linear independence. If we have vectors

$$\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n,$$

We check for independence by:

1. Constructing the matrix $A = [\mathbf{v}_1 \ \mathbf{v}_2 \ \dots \ \mathbf{v}_n]$ having the vectors as columns.
2. Reduce A to reduced row echelon form.
3. If every column has a pivot element we conclude that the vectors are linearly independent. Otherwise they are linearly dependent

Example 1.1.4

Are these vectors linearly dependent?

$$\mathbf{v}_1 = \begin{bmatrix} 5 \\ 10 \\ 5 \\ 0 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 7 \\ 5 \\ 4 \end{bmatrix} \quad \text{og} \quad \mathbf{v}_3 = \begin{bmatrix} 2 \\ 6 \\ 6 \\ 8 \end{bmatrix}$$

We reduce the matrix $[\mathbf{v}_1 \ \mathbf{v}_2 \ \mathbf{v}_3]$:

$$\begin{bmatrix} 5 & 3 & 2 \\ 10 & 7 & 6 \\ 5 & 5 & 6 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last column do not contain a pivot element. Hence $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

Example 1.1.5

Are these vectors linearly dependent?

$$\mathbf{v}_1 = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 14 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix}$$

We could reduce the following matrix:

$$\begin{bmatrix} 8 & 14 & 3 & 7 \\ 7 & -2 & 1 & 5 \\ 4 & 5 & 0 & 11 \end{bmatrix}$$

However, we do not need to in this case. We observe that we can't get more than three pivot elements. Thus there can't be a pivot element in all the four columns. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are linearly dependent.

Theorem 1.1.2

Given n vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in $\mathbb{R}^m$. If

1. one of the vectors is a linear combination of the other, or
2. $n > m$,

then the vectors are linearly dependent.

Theorem 1.1.3

If we have n vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n$ in $\mathbb{R}^n$, then they are linearly independent if

and only if they span the whole of $\mathbb{R}^n$, that is if and only if

$$\text{span}\{\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_n\} = \mathbb{R}^n.$$

Proof. Let

$$A = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \dots & \mathbf{v}_n \end{bmatrix}$$

be the $n \times n$ -matrix with our vectors as columns. We know that the vectors are linearly independent if and only if we obtain a pivot element in every column when we reduce A . Now, since A is a square matrix this is the same as having pivot elements in every row. That is equivalent with

$$A\mathbf{x} = \mathbf{b}$$

having solutions for all vectors $\mathbf{b}$ in $\mathbb{R}^n$, which says that the columns of A span $\mathbb{R}^n$. $\square$

1.2 Subspaces

We have until now been talking about the **Euclidean vector spaces** $\mathbb{R}^n$, but there also exist a whole lot more vector spaces. For example, if we instead of looking at the product of real number lines

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \dots \times \mathbb{R}}_{n \text{ copies}}$$

look at the product of complex number lines

$$\mathbb{C}^n = \underbrace{\mathbb{C} \times \mathbb{C} \times \dots \times \mathbb{C}}_{n \text{ copies}}$$

we obtain a new vector space $\mathbb{C}^n$ consisting of vectors with complex entries

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_n + ib_n \end{bmatrix}.$$

Here the scalar multiplication is given component wise with complex numbers and addition of vectors is still defined component wise. More abstract examples of vector spaces and the defining properties of them can be found in section ??.

For now we will be interested in a more nicely behaved kind of vector space, namely **subspaces** of Euclidean vector spaces.

i Definition 1.2.1

A **subspace** of $\mathbb{R}^n$ is a subset of vectors $U \subseteq \mathbb{R}^n$ such that

1. The zero vector is contained in U ,

$$\mathbf{0} \in U.$$

2. U is closed under scalar multiplication, i.e. if $\mathbf{u}$ is a vector in U and a is any scalar, then $a\mathbf{u}$ also lies in U ,

$$\mathbf{u} \in U \text{ and } a \in \mathbb{R} \implies a\mathbf{u} \in U.$$

3. U is closed under vector addition, i.e. if $\mathbf{u}$ and $\mathbf{v}$ are vectors in U , then their sum $\mathbf{u} + \mathbf{v}$ is also in U ,

$$\mathbf{u}, \mathbf{v} \in U \implies \mathbf{u} + \mathbf{v} \in U.$$

We have already seen examples of such subspaces. The linear span of a set of vectors is a subspace.

Example 1.2.1

Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_t$ be t vectors in $\mathbb{R}^n$, then

$$\text{span}(\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_t) = \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \dots + a_t\mathbf{x}_t \mid a_1, \dots, a_t \in \mathbb{R}\}$$

is a subspace.

In the last section we studied linear dependency. As a continuation of that we will now define the dimension of a subspace.

i Definition 1.2.2

The **dimension** of a subspace U , denoted $\dim(U)$, is the maximum number of linearly independent vectors in U .

That is to say, if there exists n vectors $\mathbf{v}_1, \dots, \mathbf{v}_n \in U$ that are linearly independent, but $n + 1$ vectors $\mathbf{v}_1, \dots, \mathbf{v}_{n+1} \in U$ are always linearly dependent, then the dimension $\dim U = n$. The dimension of $\{0\}$ is 0.

A **basis** of U is a maximal (dimension n) set of linearly independent vectors. If a set $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ is a basis of U , then any vector in U can be represented as a unique linear combination of this basis. i.e. for any $\mathbf{u} \in U$ there exists unique

scalars $\alpha_1, \dots, \alpha_n \in \mathbb{R}$ such that

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \dots + \alpha_n \mathbf{v}_n.$$

Example 1.2.2

The vector space $\mathbb{R}^2$ has dimension 2 and the two vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

gives a basis of $\mathbb{R}^2$.

The vector space $\mathbb{R}^3$ has dimension 3 and the three vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

gives a basis of $\mathbb{R}^3$.

The vector space $\mathbb{R}^n$ has dimension n and one basis is

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right)$$

The bases given above are called the standard bases of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$.

Subspaces of a matrix

Definition 1.2.3

The **null space** of A is the space of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$:

$$\text{Null}A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

Definition 1.2.4

The **column space** of a real $m \times n$ -matrix

$$A = [\mathbf{a}_1 \quad \mathbf{a}_2 \quad \cdots \quad \mathbf{a}_n]$$

is the subspace of $\mathbb{R}^m$ spanned by the columns of A :

$$\text{Col}A = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n\}$$

The column space of A is the collection of all linear combination of the columns in A . A product $A\mathbf{v}$ of the matrix A and a vector $\mathbf{v}$ in $\mathbb{R}^n$ is exactly a linear combination of the columns in A , so we can describe the column space as all vectors on the form $A\mathbf{v}$:

$$\text{Col}A = \{A\mathbf{v} \mid \mathbf{v} \in \mathbb{R}^n\}$$

i Definition 1.2.5

The **row space** of a real $m \times n$ -matrix

$$A = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}$$

is the subspace of $\mathbb{R}^n$ spanned by the rows in A :

$$\text{Row}A = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \dots, \mathbf{r}_m\}$$

The row space of A is all the linear combination of the rows in A (considering the rows as columns). This is equal to the column space of the transposed matrix:

$$\text{Row}A = \text{Col}A^T$$

Example 1.2.3

Let A be the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & 3 & 7 & 1 \\ 1 & 2 & 2 & 5 & 1 \\ 3 & 6 & 6 & 15 & 3 \end{bmatrix}$$

Let us try to describe the null-space, column space and row space of A .

To find the null-space, we solve the equation

$$A\mathbf{x} = \mathbf{0}.$$

We do that by reducing A :

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & 3 & 7 & 1 \\ 1 & 2 & 2 & 5 & 1 \\ 3 & 6 & 6 & 15 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtain three free variables, and the general solution is:

$$\mathbf{x} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\text{Null}A = \text{span} \left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

The vectors above is also linearly independent, so they form a basis for the null space

Directly from the definitions we have that the column and row space is given as the following sets:

$$\begin{aligned} \text{Col}A &= \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 5 \\ 15 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right) \\ \text{Row}A &= \text{span} \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 6 \\ 15 \\ 3 \end{bmatrix} \right) \end{aligned}$$

However, we might like to have basis and not only a spanning set. It turns out that we can find these bases by reducing A .

We start by considering the column space. In the reduced row echelon form of A , we have pivot elements in the first and third column. If we switch around the columns in A such that the first and third column is placed in front and do the corresponding switches in the reduced row form, we get:

$$\left[\begin{array}{cc|cc} 1 & 1 & 2 & 2 & 0 \\ 2 & 3 & 4 & 7 & 1 \\ 1 & 2 & 2 & 5 & 1 \\ 3 & 6 & 6 & 15 & 3 \end{array} \right] \sim \left[\begin{array}{cc|cc} 1 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array} \right]$$

Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$. The reduced row form now tells us that the columns without pivot elements, that is $\mathbf{a}_2, \mathbf{a}_4$ and $\mathbf{a}_5$, can be written as linear combinations of the pivot columns, $\mathbf{a}_1$ og $\mathbf{a}_3$:

$$\mathbf{a}_2 = 2 \cdot \mathbf{a}_1, \quad \mathbf{a}_4 = (-1) \cdot \mathbf{a}_1 + 3\mathbf{a}_3, \quad \mathbf{a}_5 = (-1) \cdot \mathbf{a}_1 + 1 \cdot \mathbf{a}_3.$$

Specifically, the columns without pivot elements is in the subspace spanned by the pivot columns,

$$\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5 \in \text{span}\{\mathbf{a}_1, \mathbf{a}_3\}.$$

The space spanned by all the columns in A , that is $\text{Col}(A)$, is therefore equal to the space spanned by only $\mathbf{a}_1$ og $\mathbf{a}_3$. In addition, the reduced row form of A shows that $\mathbf{a}_1$ and $\mathbf{a}_3$ are linearly independent.

This shows that

$$(\mathbf{a}_1, \mathbf{a}_3) = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 6 \end{bmatrix} \right)$$

is a basis of the column space $\text{Col}A$.

It is somewhat easier to realize how to find a basis of the row space. We observe that if we perform an elementary row operation on a matrix, we still have the same row space. This follows from the fact that the rows in the resulting matrix from a row operation are linear combinations of the rows in the old matrix. Also, we can always find a row operation that gives us the old matrix from the new one.

Hence, to describe the row space of A it is enough to consider the row reduced form of A . Here we can easily observe that every non-zero row is necessarily linearly

independent. We have that

$$\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right)$$

is a basis of the row space $\text{Row}A$.

Remark 1.2.1

The example above shows us a method of constructing a basis for the column space of a matrix A :

1. Reduce A to reduced row echelon form.
2. Identify the pivot columns.
3. The pivot columns gives a basis for $\text{Col}A$.

Warning: Remember that in the last step we need to use the columns in the original matrix A . Row operations do change the column space.

Theorem 1.2.1

Let A be an $m \times n$ -matrix, and let E be the reduced row form of A . Then:

- (a) The dimension of the null space, $\text{Null}A$, is equal to the number of free variables we get when solving $A\mathbf{x} = \mathbf{0}$, which is equal to the amount of columns without pivot elements in E .
- (b) The dimension of the column space, $\text{Col}A$, is equal to the amount of columns containing pivot elements in E .
- (c) The dimension of the row space, $\text{Row}(A)$, equals the amount of rows not equal to zero in E .

Theorem 1.2.2

Let A be a matrix. The column and row space of A has equal dimension

$$\dim \text{Col}A = \dim \text{Row}A$$

This number, which is the dimension of both the column and row space is called the **rank** of A .

$$\text{rank}A = \dim \text{Col}A = \dim \text{Row}A$$

Since every column in the reduced row form either contains a pivot element or gives a free variable for the equation $A\mathbf{x} = \mathbf{0}$, we obtain the following result from theorem 1.2.1.

Theorem 1.2.3

Let A be a $m \times n$ -matrix. Then

$$\dim \text{Null}A + \text{rank}A = n.$$

Let us end this section by combining these results with results on invertability

Theorem 1.2.4

Let A be a $n \times n$ -matrix. The following statements are equivalent

1. A is invertible
2. The columns of A are linearly independent
3. The rows of A are linearly independent
4. $\text{rank}A = n$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution ($\mathbf{x} = A^{-1}\mathbf{b}$)
6. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{c}$.
7. The reduced row echelon form of A is I_n
8. A has a pivot element in every row
9. $A\mathbf{x} = \mathbf{0}$ has only one solution, $\mathbf{x} = \mathbf{0}$
10. $\text{Col}(A) = \mathbb{R}^n$
11. $\text{Row}(A) = \mathbb{R}^n$

1.3 Determinant

One of the sought after properties of a square matrix ($n \times n$) is invertability. If a matrix A is invertible, then we know that every equation $A\mathbf{x} = \mathbf{b}$ has a unique

solution $x = A^{-1}\mathbf{b}$. We will now explore a number which determines invertability, namely **the determinant**. If we have $n \times n$ -matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

then the determinant of A is denoted by either

$$\det A \quad \text{or} \quad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Determinants of 2×2 -matrices

The determinant of a 2×2 -matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 1.3.1

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 9 = -5$$

Determinants of 3×3 -matrices

For a 3×3 -matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

we define the determinant to be:

$$\begin{aligned}\det A &= \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} \\ &= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}\end{aligned}$$

We will present the general rule for arbitrary large matrices in a bit, but let us observe that the rules at the moment are:

- the sign switches as $+ - + - + -$ horizontally (addition first)
- the smaller "sub-determinants" are obtained by deleting the row and column which the coefficient of it lie in, i.e.

the "sub-determinant"

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

is found by deleting the row and column containing a_{11} :

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

likewise, we obtain

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

through deleting the first row and second column:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and lastly

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

is obtained by deleting the first row and third column:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example 1.3.2

$$\begin{aligned} \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} &= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1(45 - 48) - 2(36 - 42) + 3(32 - 35) \\ &= -3 + 12 - 9 = 0 \end{aligned}$$

Determinants: general definition

We define the determinant of an arbitrary big quadratic matrix in the same pattern as the determinant of a 3×3 -matrix.

Definition 1.3.1

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be an $n \times n$ -matrix. The **determinant** of A is defined as follows.

1. If $n = 1$, then $A = [a_{11}]$, and we define $\det A = a_{11}$.
2. If $n > 1$, we introduce a few helping variables. For every i and j from 1 to n we set A_{ij} to be the $(n-1) \times (n-1)$ -matrix we obtain by removing row i and column j from A , and we define

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

to be the determinant of this matrix with a sign dependent on i and j . The

determinant of A is defined as:

$$\det A = \sum_{j=1}^n a_{1j}C_{1j}$$

The numbers C_{ij} in the definition are called **cofactors** of A .

One can now see that this definition reduces back to our already defined cases for 2×2 -matrices and 3×3 -matrices.

Example 1.3.3

Let A be the following 4×4 -matrix:

$$A = \begin{bmatrix} 3 & 0 & 2 & 4 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 5 & 3 \end{bmatrix}$$

We calculate the determinant of A . From the definition we have:

$$\begin{aligned} \det A &= 3 \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} \\ &\quad + 2 \begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 7 & 0 & 3 \end{vmatrix} - 4 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 7 & 0 & 5 \end{vmatrix} \end{aligned}$$

We calculate each of the 3×3 -determinants needed (Note that we do not calculate the second, since it will be multiplied by zero):

$$\begin{aligned} \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 3 \end{vmatrix} &= 2 \begin{vmatrix} 1 & 0 \\ 5 & 3 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 0 & 5 \end{vmatrix} = 6 \\ \begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 7 & 0 & 3 \end{vmatrix} &= 1 \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ 7 & 3 \end{vmatrix} + 0 \begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} = -12 \\ \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 7 & 0 & 5 \end{vmatrix} &= 1 \begin{vmatrix} 0 & 1 \\ 0 & 5 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 7 & 5 \end{vmatrix} + 1 \begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} = -6 \end{aligned}$$

Substituting these values into the expression for $\det A$ we have:

$$\det A = 3 \cdot 6 - 0 + 2 \cdot (-12) - 4 \cdot (-6) = 18$$

Cofactor-expansion

In the definition of the determinant we move through the first row in the matrix and look at the numbers

$$a_{11}, a_{12}, \dots, a_{1n}.$$

Each number a_{1j} is multiplied with the associated cofactor C_{1j} , and at the end we add all these products

However, we can also move along any other row with the same procedure and get the same end product. Also, we can move along any column.

Theorem 1.3.1

Let A be a $n \times n$ -matrix, where $n > 1$, and let A_{ij} and C_{ij} be given as in the definition of the determinant. Then we have

$$\det A = \sum_{j=1}^n a_{kj} C_{kj} = \sum_{i=1}^n a_{il} C_{il}$$

for all k and l such that $1 \leq k \leq n$ and $1 \leq l \leq n$.

The method of finding the determinant through cofactors as in the theorem is called **cofactorexpansion**. We say that we do the cofactor-expansion along row k or along column l .

Let us now find the same determinant as in example 1.3.3, but in a more clever way.

We have to make sure that the signs in the cofactors are correct. When we expand along the first row (as in the definition), or along the first column, we always start with a positive sign in the first part. However, when we expand along a different row or column, it might be that we start with a negative sign. A good rule of thumb is to use the following diagram to remember which sign each cofactor has:

$$\begin{bmatrix} + & - & + & - & \dots \\ - & + & - & + & \dots \\ + & - & + & - & \dots \\ - & + & - & + & \dots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 1.3.4

Once again, let A be the following 4×4 -matrix:

$$A = \begin{bmatrix} 3 & 0 & 2 & 4 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 5 & 3 \end{bmatrix}$$

We observe that the second column consists almost entirely of zeros, so it can be smart to do the expansion along it. This gives us:

$$\begin{aligned} \det A &= -0 \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} + 2 \begin{vmatrix} 3 & 2 & 4 \\ 2 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} \\ &\quad - 0 \begin{vmatrix} 3 & 2 & 4 \\ 1 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} + 0 \begin{vmatrix} 3 & 2 & 4 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} \\ &= 2 \cdot \left(4 \cdot \begin{vmatrix} 2 & 1 \\ 7 & 5 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \right) \\ &= 2 \cdot (4 \cdot 3 + 3 \cdot (-1)) = 18 \end{aligned}$$

This is the same result as in example 1.3.3, but with less work, since we only needed to find one 3×3 -determinant.

Determinants and row operations

When doing row operations on a matrix, we get a new matrix. This matrix doesn't necessarily have the same determinant as the original. However, it turns out that the determinant changes in controlled ways under row operations. We can exploit this to ease our calculations.

Theorem 1.3.2

Let A be a $n \times n$ -matrix, and let B be matrix we get by doing a row operation on A . Then we have the following correspondance between the determinant of A

and B , based on which type of row operation done:

Row operation	Result
Multiplication by a number k	$\det B = k \cdot \det A$
Add a multiple of a row to another	$\det B = \det A$
exchange two rows	$\det B = -\det A$

Example 1.3.5

We find $\det A$, where

$$A = \begin{bmatrix} 3 & 3 & 12 \\ 2 & 2 & 13 \\ 4 & 2 & 19 \end{bmatrix}$$

$$\begin{aligned} \det A &= \begin{vmatrix} 3 & 3 & 12 \\ 2 & 2 & 13 \\ 4 & 2 & 19 \end{vmatrix} = 3 \cdot \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 13 \\ 4 & 2 & 19 \end{vmatrix} \\ &= 3 \cdot \begin{vmatrix} 1 & 1 & 4 \\ 0 & 0 & 5 \\ 0 & -2 & 3 \end{vmatrix} = -3 \cdot \begin{vmatrix} 1 & 1 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{vmatrix} \\ &= -3 \cdot 1 \cdot \begin{vmatrix} -2 & 3 \\ 0 & 5 \end{vmatrix} \\ &= -3 \cdot 1 \cdot (-2) \cdot 5 = 30 \end{aligned}$$

Here we started by doing row operations on the matrix, while simultaneously keeping track of how the determinant changed.

First we multiplied the first row with $1/3$. This made the determinant of the new matrix $1/3$ the size of the determinant of A , so we had to multiply it with 3 for the numbers to stay equal

After that we subtracted a multiple of the first row from the two other rows, which resulted in no change of the determinant.

Then we exchanged the two bottom rows, which made the determinant change sign.

At the end we expanded along the first column.

Triangulære matriser

We say that a $n \times n$ -matrix is **upper triangular** if all the numbers below the diagonal are 0, that is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Similarly, a $n \times n$ -matrix is **lower triangular** if all numbers above the diagonal are 0, that is:

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Example 1.3.6

In example 1.3.5 we used row operations to get our matrix on the form:

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

This matrix is upper triangular.

Theorem 1.3.3

Let A be a (upper or lower) triangular $n \times n$ -matrix. Then the determinant of A is equal to the product of the numbers along the diagonal of A :

$$\det A = a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn}$$

Example 1.3.7

$$\begin{vmatrix} 1 & 1 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{vmatrix} = 1 \cdot (-2) \cdot 5$$

A good strategy when calculating the determinant of a big and unpleasant matrix, is to reduce it to an upper triangular form (and keep track of how the determinant changes with the help of Theorem 1.3.2), and then calculate the determinant of the triangular matrix using Theorem 1.3.3.

Some nice results on calculation

Theorem 1.3.4

The determinant of a product of two matrices is equal to the product of the determinants. That is: If A and B are two $n \times n$ -matrices, then

$$\det(AB) = (\det A)(\det B).$$

Theorem 1.3.5

The determinant do not change under transposition. That is: If A is a $n \times n$ -matrix, then

$$\det A = \det A^T.$$

Karakterisering av inverterbarhet

We can also prove that a square matrix is invertible if and only if it has a nonzero determinant, thus we can expand Theorem 1.2.4:

Theorem 1.3.6

Let A be a $n \times n$ -matrix. The following statements are equivalent

1. A is invertible
2. $\det A \neq 0$
3. The columns of A are linearly independent
4. The rows of A are linearly independent
5. $\text{rank} A = n$

6. $A\mathbf{x} = \mathbf{b}$ has a unique solution ($\mathbf{x} = A^{-1}\mathbf{b}$)
7. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{c}$.
8. The reduced row echelon form of A is I_n
9. A has a pivot element in every row
10. $A\mathbf{x} = \mathbf{0}$ has only one solution, $\mathbf{x} = \mathbf{0}$
11. $\text{Col}(A) = \mathbb{R}^n$
12. $\text{Row}(A) = \mathbb{R}^n$