Problems for fifth day

1. Calculate the eigenvalues and assicated eigenspaces of

a)

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

b)

$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

2. Calculate the eigenvalues and eigenvectors of

a)

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

b)

$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

c)

$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

d)

$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

3. Let the eigenvectors of

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

be the columns of a 3×3-matrix P and calculate $P^{-1}AP$.

4. Find a formula for A^n and calculate A^{10} when

a)

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

5. Find a 3×3-matrix which has eigenvalues 1, 2 and 3, with associated eigenvectors

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} \quad \text{og} \quad \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}.$$

Solutions for fifth day

1. a In order to find the eigenvalues of a matrix A we have to find the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$, where A is a quadratic matrix and I is the identity matrix of same size.

In this problem, we get, if the matrix is denoted by A, that

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix} - \begin{bmatrix}\lambda & 0\\0 & \lambda\end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix}1-\lambda & 0\\0 & -\lambda\end{bmatrix}\right) \\ &= \lambda(\lambda - 1) \end{aligned}$$

The equation have the solutions $\lambda_1 = 0$ and $\lambda_2 = 1$. These are the eigenvalues of the matrix.

The eigenvectors of A are associated to the eigenvalues, and are found as the non-zero vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ such that

$$A\mathbf{x}_1 = \lambda_1\mathbf{x}_1 \text{ and } A\mathbf{x}_2 = \lambda_2\mathbf{x}_2.$$

We observe that any vector on the form

$$\mathbf{x}_1 = \begin{bmatrix}0\\a\end{bmatrix}$$

gives

$$A\mathbf{x}_1 = 0\mathbf{x}_1 = \mathbf{0}.$$

The vector $\mathbf{x}_1 = \begin{bmatrix}0\\1\end{bmatrix}$ is therefore an eigenvector of λ_1.

We also observe that any vector on the form

$$\mathbf{x}_2 = \begin{bmatrix}a\\0\end{bmatrix}$$

gives

$$A\mathbf{x}_2 = 1\mathbf{x}_2 = \mathbf{x}_2.$$

The vector $\mathbf{x}_2 = \begin{bmatrix}1\\0\end{bmatrix}$ is therefore an eigenvector of λ_2.

The eigenspace associates to λ_1 is therefore the y-axis, and the eigenspace of λ_2 is the x-axis.

1.b We find the characteristic polynomial:

$$\begin{aligned} p(\lambda) &= \det\left(\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)\\ &= \det\left(\frac{1}{2}\begin{bmatrix}1-2\lambda & 1\\ 1 & 1-2\lambda\end{bmatrix}\right)\\ &= \frac{1}{2}((1-2\lambda)(1-2\lambda)-1)\\ &= 2\lambda(\lambda-1)\end{aligned}$$

The equation $p(\lambda) = 0$ have the solutions $\lambda_1 = 0$ and $\lambda_2 = 1$. These are the eigenvalues of the matrix.

We observe that $\mathbf{x}_1 = \begin{bmatrix}-1\\ 1\end{bmatrix}$ solves

$$A\mathbf{x}_1 = 0\mathbf{x}_1 = \mathbf{0}.$$

The vector $\mathbf{x}_1 = \begin{bmatrix}-1\\ 1\end{bmatrix}$ is therefore an eigenvector associated to $\lambda_1 = 0$.

$\mathbf{x}_2$ must lie in the null space of

$$A - I = \frac{1}{2}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}.$$

This is spanned by the vector $\begin{bmatrix}1\\ 1\end{bmatrix}$, so we choose

$$\mathbf{x}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}.$$

And verify that $A\mathbf{x}_2 = 1\mathbf{x}_2 = \mathbf{x}_2$. In this case, the eigenspace of λ_1 is the line spanned by $\begin{bmatrix}-1\\ 1\end{bmatrix}$, and the eigenspace of λ_2 is the line spanned by $\begin{bmatrix}1\\ 1\end{bmatrix}$.

2. a Eigenvalues are $\lambda_1 = -2, \lambda_2 = i$ and $\lambda_3 = -i$.

The eigenspaces are spanned by the corresponding eigenvectors

$$\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}i\\ 1\\ 1\end{bmatrix}, \begin{bmatrix}-i\\ 1\\ 1\end{bmatrix}$$

2. b

$$\det(A - \lambda I) = (\lambda + 1)^2(\lambda - 2)$$

gives eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = 2$$

The eigenspaces are spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

3. We found the eigenvalues and eigenvectors of the matrix in problem 2a.

$$P = \begin{bmatrix} 0 & i & -i \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -2 \\ -i & 0 & 1 \\ i & 0 & 1 \end{bmatrix}$$

Calcualted

$$P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

Which expectedly is a diagonal matrix where the elements along the diagonal are the eigenvalues of A.

4. a The eigenvalues of A is 3 and -5 with eigenvectors respectively $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. We therefore define

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \text{ og } P = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$$

and find

$$P^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}.$$

Now, we know $A = PDP^{-1}$, and therefore $A^k = PD^kP^{-1}$. We express this last product explicitly :

$$\begin{aligned} A^k &= \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & (-5)^k \end{bmatrix}\frac{1}{8}\begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix} \\ &= \frac{1}{8}\begin{bmatrix} 6 \cdot 3^k + 2 \cdot (-5)^k & 3 \cdot 3^k - 3(-5)^k \\ 4 \cdot 3^k - 4 \cdot (-5)^k & 2 \cdot 3^k + 6 \cdot (-5)^k \end{bmatrix}. \end{aligned}$$

For $k = 10$ we have

$$A^{10} = \begin{bmatrix} 2485693 & -3639966 \\ -4853288 & 7338981 \end{bmatrix}.$$

4. b A is upper triangular, so the eigenvalues are the numbers along the diagonal of A: 2, 3 and 5. We make the matrix P and D:

$$P = \begin{bmatrix} 1 & 3 & 25 \\ 0 & 1 & 15 \\ 0 & 0 & 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

We use row recution, and the fact that $[B|I] \sim [I|C]$, means $B^{-1} = C$ to find

$$P^{-1} = \begin{bmatrix} 1 & -3 & 10/3 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1/6 \end{bmatrix}$$

These reduction steps are given as:

$$\begin{bmatrix} 1 & 3 & 25 & 1 & 0 & 0 \\ 0 & 1 & 15 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & -\frac{25}{6} \\ 0 & 1 & 0 & 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 6 & 0 & 0 & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 & 1 & -3 & \frac{10}{3} \\ 0 & 1 & 0 & 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{6} \end{bmatrix}$$

thus we have diagonalized the matrix A.

Notice that when $A = PDP^{-1}$, we have

$$\begin{aligned} A^2 &= (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} \\ &= PD^2P^{-1}, \\ A^3 &= (PD^2P^{-1})(PDP^{-1}) = PD^2(P^{-1}P)DP^{-1} \\ &= PD^3P^{-1}, \end{aligned}$$

and so on. In general:

$$A^n = PD^nP^{-1}$$

For $n = 10$:

$$\begin{aligned} A^{10} &= PD^{10}P^{-1} \\ &= \begin{bmatrix} 1 & 3 & 25 \\ 0 & 1 & 15 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 3^{10} & 0 \\ 0 & 0 & 5^{10} \end{bmatrix} \begin{bmatrix} 1 & -3 & 10/3 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1/6 \end{bmatrix} \\ &= \begin{bmatrix} 1024 & 174075 & 40250650 \\ 0 & 59049 & 24266440 \\ 0 & 0 & 9765625 \end{bmatrix} \end{aligned}$$

5. Construct a diagonal matrix D with the eigenvalues along the diagonal, and a matrix V with the eigenvectors as columns:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{og} \quad V = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 4 \\ 2 & 5 & 2 \end{bmatrix}$$

Then $A = VDV^{-1}$ is our wanted matrix. We find the inverse matrix of V:

$$V^{-1} = \begin{bmatrix} 8 & -1 & -2 \\ -2 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}$$

Now we multiply the matrices and end up with:

$$A = VDV^{-1} = \begin{bmatrix} -9 & 2 & 2 \\ -36 & 9 & 6 \\ -22 & 4 & 6 \end{bmatrix}$$