Problems for seventh day

1. Find e^{At} when

a)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

ab

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

d)

$$A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

2. Find the solutions of the initial value problems

a)

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

b)

$$\mathbf{y}' = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

d)

$$\mathbf{y}' = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

3. Find e^{At} when

a)

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

4. Find the general solution of $A\mathbf{y} = \mathbf{y}'$, when

a)

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} -6 & -11 & 16 \\ 2 & 5 & -4 \\ -4 & -5 & 10 \end{bmatrix}$$

Solutions for seventh day

1. a We recall from yesterday that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is diagonalizable as

$$A = PDP^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^0 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

1. b We recall from yesterday that

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is diagonalizable as

$$A = PDP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Thus

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1+e^t & -1+e^t \\ -1+e^t & 1+e^t \end{bmatrix} \end{aligned}$$

1. c We recognize A as being a Jordan block, so we now that

$$A = 2I_2 + N = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

And since $2I_2$ and N commute ($2I_2N = 2N = N2 = NI_22 = N(2I_2)$), we have

$$e^{At} = e^{2I_2t}e^{Nt}.$$

Now,

$$e^{2I_2t} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} = e^{2t}I_2$$

and

$$\begin{aligned} e^{Nt} &= I_2 + \frac{t}{1!}N + \frac{t^2}{2!}N^2 + \frac{t^3}{3!}N^3 + \cdots \\ &= \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{t}{1!}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix} + \cdots = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + \frac{t}{1!}\begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix} \\ &= \begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix} \end{aligned}$$

gives us

$$e^{At} = e^{2t}\begin{bmatrix}1 & t\\ 0 & 1\end{bmatrix}$$

1. d We start by finding the eigenvalues of A.

$$\begin{aligned} \rho_A(\lambda) = \begin{vmatrix}\frac{1}{\sqrt{2}} - \lambda & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \lambda\end{vmatrix} &= (\frac{1}{\sqrt{2} - \lambda})^2 + \frac{1}{2} \\ &= \frac{1}{2} - \frac{2}{\sqrt{2}} + \lambda^2 + \frac{1}{2} \\ &= \lambda^2 - \sqrt{2}\lambda + 1 \end{aligned}$$

$$\lambda = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$$

We have complex eigenvalues, so we try to factor A as PCP^{-1}, where

$$P = \begin{bmatrix}\text{Re}\mathbf{v} & \text{Im}\mathbf{v}\end{bmatrix}$$

for any eigenvector $\mathbf{v}$ of A, and

$$C = \begin{bmatrix}a & -b\\ b & a\end{bmatrix}$$

where $\lambda = a \pm ib = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$. The observant reader may already have figured out that $A = C$, and we could skip past finding the eigenvectors. For the not so observant reader we head on with seemingly non-interesting calculations for the eigenvector.

The eigenspace of $\lambda = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ is found as

$$E(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \text{Null}\left(\begin{bmatrix}i\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}}\\ \frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}}\end{bmatrix}\right) = \text{Null}\left(\begin{bmatrix}i & -1\\ 0 & 0\end{bmatrix}\right) = \text{span}\left(\begin{bmatrix}1\\ i\end{bmatrix}\right)$$

We therefore have that $\begin{bmatrix}1\\ i\end{bmatrix}$ is an eigenvector, and since $\text{Re}\mathbf{v} = \begin{bmatrix}1\\ 0\end{bmatrix}$ and $\text{Im}\mathbf{v} = \begin{bmatrix}0\\ 1\end{bmatrix}$, we see that

$$P = \begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} = I_2$$

and

$$A = PCP^{-1} = I_2 C I_2 = C$$

which the observant reader saw a few unnecessary calculations ago.

Now, we rewrite A as

$$A = \frac{1}{\sqrt{2}} I_2 + \frac{1}{\sqrt{2}} S, \text{ for } S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and get

$$e^{At} = e^{t/\sqrt{2}} e^{1/\sqrt{2}\ S} = e^{t/\sqrt{2}} \begin{bmatrix} \cos(t/\sqrt{2}) & -\sin(t/\sqrt{2}) \\ \sin(t/\sqrt{2}) & \cos(t/\sqrt{2}) \end{bmatrix}$$

2. We know that the solution of the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{b}$$

is given by

$$\mathbf{y} = e^{At}\mathbf{b}.$$

From the first problem we therefore read out the solutions:

a)

$$\mathbf{y} = e^{At} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^t \\ 2 \end{bmatrix}$$

b)

$$\mathbf{y} = e^{At} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

c)

$$\mathbf{y} = e^{At} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 + 4t \\ 4 \end{bmatrix}$$

d)

$$\mathbf{y} = e^{At} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4e^{t/\sqrt{2}} \begin{bmatrix} \cos(t/\sqrt{2}) - \sin(t/\sqrt{2}) \\ \cos(t/\sqrt{2}) + \sin(t/\sqrt{2}) \end{bmatrix}$$

3. a In the lecture yesterday, we found that

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$$

can be diagonalized as

$$A = PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \frac{1}{-8} \begin{bmatrix} -2 & -1 \\ -2 & 3 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}.$$

Thus

$$\begin{aligned} e^{At} &= \frac{1}{8}\begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-5t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix} \\ &= \frac{1}{8}\begin{bmatrix} 3e^{3t} & e^{-5t} \\ 2e^{3t} & -2e^{-5t} \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix} \\ &= \frac{1}{8}\begin{bmatrix} 6e^{3t} + 2e^{-5t} & 3e^{3t} - 3e^{-5t} \\ 4e^{3t} - 4e^{-5t} & 2e^{3t} + 6e^{-5t} \end{bmatrix} \end{aligned}$$

3. b In yesterdays exercises we found the diagonalization

$$\begin{aligned} A &= PDP^{-1} \\ &= \begin{bmatrix} 1 & 3 & 25 \\ 0 & 1 & 15 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}\begin{bmatrix} 1 & -3 & 10/3 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1/6 \end{bmatrix} \end{aligned}$$

Thus we have

$$e^{At} = \begin{bmatrix} 1 & 3 & 25 \\ 0 & 1 & 15 \\ 0 & 0 & 6 \end{bmatrix}\begin{bmatrix} e^{2t} & 0 & 0 \\ 0 & e^{3t} & 0 \\ 0 & 0 & e^{5t} \end{bmatrix}\begin{bmatrix} 1 & -3 & 10/3 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1/6 \end{bmatrix}$$

We could be satisfied with this expression and go on with our day, or we could insist on calculating the product. You might want to let a computer do it for you, but in any case you should get

$$e^{At} = \begin{bmatrix} e^{2t} & 3e^{3t} - 3e^{2t} & e^{2t} \cdot 10/3 + 3e^{3t}(-5/2) + 25e^{5t} \cdot 1/6 \\ 0 & e^{3t} & e^{3t}(-5/2) + 15e^{5t} \cdot 1/6 \\ 0 & 0 & 6e^{5t} \cdot 1/6 \end{bmatrix}.$$

4. a We found in yesterdays exercises that A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$. The eigenspace of λ_1 is spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \text{ and } \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

and the eigenspace of λ_2 is spanned by

$$\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

The general solution of $A\mathbf{y} = \mathbf{y}'$ is therefore given by

$$\mathbf{y} = c_1 e^{-t}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{2t}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

4. b The eigenvalues of A is $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 4$, and we can find corresponding eigenvectors as

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and } \mathbf{v}_3 = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}.$$

A general solution is therefore

$$\mathbf{y} = c_1 e^{2t} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}.$$