Chapter 1

Systems of differential equations

Let us now step back to ODEs. In many applied problems, several quantities are varying continuously in time, and can be described through a **system of differential equations**[1]

$$\begin{aligned}
y_1'(t) &= a_{11}y_1(t) + a_{12}y_2(t) + \cdots + a_{1n}y_n(t) \\
y_2'(t) &= a_{21}y_1(t) + a_{22}y_2(t) + \cdots + a_{2n}y_n(t) \\
&\vdots \\
y_n'(t) &= a_{n1}y_1(t) + a_{n2}y_2(t) + \cdots + a_{nn}y_n(t)
\end{aligned}$$

Here $y_1, \ldots, y_n$ are differentiable functions of t, with derivatives $y_1', \ldots, y_n'$. The a_{ij} are constants. In our discussion we will assume that the constants a_{ij} are real numbers. Alternatively, we can write this as

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}.$$

We also use the short hand

$$\mathbf{y}' = A\mathbf{y}.$$

A solution of this is a vector of functions that satisfies the equation above. Both derivation and matrix multiplication is linear, so we rediscover the superposition principle we already saw in the case of a single ODE. That is:

[1] Note that we have implicitly assumed in the following that our coefficients are constants and that the system is homogeneous. This is however not the case in general.

Theorem 1.0.1 Superposition principle

If $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of the system

$$\mathbf{y}' = A\mathbf{y}$$

then also $c_1\mathbf{y}_1 + c_2\mathbf{y}_2$ is a solution for all real numbers c_1 and c_2.

Proof. Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two solutions.

$$\begin{aligned}
&(c_1\mathbf{y}_1 + c_2\mathbf{y}_2)' \\
=&c_1\mathbf{y}_1' + c_2\mathbf{y}_2' && \text{derivation is linear} \\
=&c_1A\mathbf{y}_1 + c_2A\mathbf{y}_2 && \mathbf{y}_1 \text{ og } \mathbf{y}_2 \text{ are solutions} \\
=&A(c_1\mathbf{y}_1 + c_2\mathbf{y}_2) && \text{Matrix multiplication is linear}
\end{aligned}$$

□

Example 1.0.1

Consider the system

$$\begin{array}{rcrcr}
y_1'(t) & = & y_1(t) & + & 2y_2(t) \\
y_2'(t) & = & 2y_1(t) & - & 2y_2(t)
\end{array}$$

which can be written as

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Two solutions of this system (verify yourself) is

$$\begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

and

$$\begin{bmatrix} e^{-3t} \\ -2e^{-3t} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-3t}$$

Initial value problem

As in the one-dimensional case there is a plethora of possible solutions to a system of ODEs. If you have a solution you can scale it arbitrary to get an infinite amount of other solutions. However, we will see that if we impose an extra condition we reduce the amount of solutions from the infinite to the singular. Such a condition is as always called an initial condition,

$$\mathbf{y}(0) = \mathbf{b}, \qquad \mathbf{b} \in \mathbb{R}^n$$

Definition 1.0.1

An initial value problem in systems of ODEs is a system

$$\mathbf{y}' = A\mathbf{y}$$

with a condition

$$\mathbf{y}(0) = \mathbf{b}$$

In the one-dimensional case we saw that $y' = ay$ $y(0) = y_0$ was solved by

$$y = y_0 e^{at}$$

and we will in fact see that our Initial value problem here will have the solution

$$\mathbf{y} = e^{At}\mathbf{b}$$

To understand this we first need to find out what the **matrix exponential**,

$$e^{At}$$

is exactly. [2] The Taylor series of the ordinary exponential e^t is

$$e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \cdots + \frac{1}{k!}t^k + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

So we naively define the matrix exponential for a $n \times n$-matrix A by substituting t in the series above with At,

$$e^A = I_n + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots + \frac{1}{k!}(At)^k + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n, \qquad \text{where } (At)^0 = I_n.$$

We will inherit some properties from the ordinary exponential, firstly we have

[2] In this note we will only scratch the surface when it comes to matrix exponentials. The interested or skeptical reader may want to consult other sources, for example the book on ODEs listed on the webpage.

Theorem 1.0.2

Let A be a $n \times n$-matrix, then

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

In addition

Theorem 1.0.3

The exponential of the zero matrix is the identity matrix,

$$e^{0_n} = I_n$$

also, for every square matrix A, the matrix exponential e^{At} is invertible and the inverse is given by

$$(e^{At})^{-1} = e^{-At}$$

However, in general the property of $e^{a+b} = e^a e^b$ doesn't generalize except for commuting matrices, that is:

Theorem 1.0.4

If $AB = BA$, then

$$Ae^B = e^B A,$$
$$e^A e^b = e^b e^A$$

and

$$e^{A+B} = e^A e^B$$

Let us now state our main result which we revealed right above.

Theorem 1.0.5 Existence and uniqueness of solution

For every $\mathbf{b} \in \mathbb{R}^n$, the initial value problem

$$\mathbf{y}' = Ay, \qquad \mathbf{y}(0) = \mathbf{b}$$

has the unique solution

$$\mathbf{y} = e^{At}\mathbf{b}$$

1.1 Calculating the matrix exponential

If we have an invertible matrix P and a matrix B such that A can be decomposed as $A = PBP^{-1}$, then we have

$$\begin{aligned}
e^{At} =& e^{PBP^{-1}t} \\
=& \sum_{n=0}^{\infty} \frac{1}{n!}(PBP^{-1}t)^n \\
=& \sum_{n=0}^{\infty} \frac{1}{n!}(P(Bt)P^{-1})^n \\
=& \sum_{n=0}^{\infty} \frac{1}{n!}P(B^n t)P^{-1} \\
=& P\left(\sum_{n=0}^{\infty} \frac{1}{n!}B^n t\right)P^{-1} \\
=& Pe^{Bt}P^{-1}.
\end{aligned}$$

also, if $B = D$ is a diagonal matrix, i.e. A is diagonalizable, then we can explicitly find e^{Dt} as

$$e^{Dt} = \begin{bmatrix} e^{d_1 t} & & \Huge 0 \\ & \ddots & \\ \Huge 0 & & e^{d_n t} \end{bmatrix}.$$

Diagonalizable

Example 1.1.1

a)Find the matrix exponential e^{At} of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

We start by finding the eigenvalues and eigenvectors of the matrix. The characteristic polynomial of A is

$$\begin{aligned}
\rho_A(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 =& \lambda^2 - 2\lambda - 3 \\
=& (\lambda - 3)(\lambda + 1)
\end{aligned}$$

so the eigenvalues of the matrix are $\lambda_1 = 3$ and $\lambda_2 = -1$. Let us now find the eigenvectors belonging to λ_1:

$$\mathrm{Null}(A - 3I_2) = \mathrm{Null}\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \mathrm{Null}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

So an eigenvector of λ_1 is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Now for the eigenvectors of λ_2:

$$\mathrm{Null}(A + 1I_2) = \mathrm{Null}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \mathrm{Null}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

So an eigenvector of λ_2 is $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

A has two linearly independent eigenvectors and is therefore diagonalizable with

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The inverse matrix P^{-1} is given by

$$P^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Since we now have that $A = PDP^{-1}$, we can calculate the exponentials using what we learnt above

$$\begin{aligned} e^{At} &= e^{PDtP^{-1}} = Pe^{Dt}P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{3t} & e^{3t} \\ e^{-t} & -e^{-t} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{3t} + e^{-t} & e^{3t} - e^{-t} \\ e^{3t} - e^{-t} & e^{3t} + e^{-t} \end{bmatrix} \end{aligned}$$

b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

We know that the solution of initial value problem is given by

$$\mathbf{y} = e^{At}\begin{bmatrix}2\\0\end{bmatrix}$$

and since we have already calculated e^{At} we have

$$\begin{aligned}\mathbf{y} &= \frac{1}{2}\begin{bmatrix}e^{3t}+e^{-t} & e^{3t}-e^{-t}\\ e^{3t}-e^{-t} & e^{3t}+e^{-t}\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}\\ &= \begin{bmatrix}e^{3t}+e^{-t}\\ e^{3t}-e^{-t}\end{bmatrix}\\ &= \begin{bmatrix}1\\1\end{bmatrix}e^{3t} + \begin{bmatrix}1\\-1\end{bmatrix}e^{-t}\end{aligned}$$

General solution when diagonalizable

Now, let us try to find a solution of

$$\mathbf{y}' = A\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{b}$$

when $A = PDP^{-1}$ is diagonalizable. Then we can do a substitution, by introducing the new vector $\mathbf{c} = P^{-1}\mathbf{b}$. The matrix P consists of linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ belonging to the eigenvalues $\lambda_1, \dots, \lambda_n$ in corresponding entries of the diagonal in D, so we have that the solution is

$$\begin{aligned}\mathbf{y} &= e^{At}\mathbf{b} = Pe^{Dt}P^{-1}\mathbf{b} = Pe^{Dt}\mathbf{c}\\ &= \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}e^{\lambda_1} & & \text{\huge 0}\\ & \ddots & \\ \text{\huge 0} & & e^{\lambda_n}\end{bmatrix}\begin{bmatrix}c_1\\ \vdots c_n\end{bmatrix}\\ &= c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2 + \cdots c_ne^{\lambda_n t}\mathbf{v}_n\end{aligned}$$

Let us summarize this in a result

Theorem 1.1.1 Diagonalizable matrices

Let A be a real valued $n \times n$ matrix with n linearly independent eigenvectors, then the general solution of

$$\mathbf{y}' = A\mathbf{y}$$

is on the form

$$c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_ne^{\lambda_n t}$$

where

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

are the linearly independent eigenvectors of A,

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

are the corresponding eigenvalues (possibly with repetition of values), and

$$c_1, c_2, \ldots, c_n$$

are scalars.

Example 1.1.2

Consider the system $\mathbf{y}' = A\mathbf{y}$ with

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

The matrix is upper triangular so we read out the eigenvalues from the diagonal as 1, 2 og 4. Eigenvectors of these are respectively (check yourself)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \text{og} \quad \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

The general solution of the system is therefore

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_4 \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} e^{4t}.$$

Jordan form

All $n \times n$-matrices A can be written on the form PJP^{-1} for some invertible matrix P and a matrix in what is called Jordan form J. When the matrix in question is diagonalizable, then J is diagonal and we can do as above. In other cases, J is nearly diagonal. Let us look at the case when J is on the form

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & \lambda & 1 & 0 & \cdots & 0 & 0 & 0 \\ & & \lambda & 1 & \cdots & 0 & 0 & 0 \\ & & & \lambda & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & \lambda & 1 & 0 \\ & & & & & & \lambda & 1 \\ & & & & & & & \lambda \end{bmatrix}$$

In particular we now know that A has the eigenvalue λ as its only eigenvalue. We can choose to write J as a sum of matrices $\lambda I_n + N$ where N is

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & & 0 & 1 & \cdots & 0 & 0 & 0 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{bmatrix}$$

Now, we observe that $e^{Jt} = e^{\lambda I_n t} e^{Nt}$, since $(\lambda I_n)N = N(\lambda I_n)$. Also, $e^{\lambda t I_n}$ is equal to $e^{\lambda t} I_n$ since λI_n is diagonal. Further, we can see that $N^n = 0$,

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ & & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$N^{n-3} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ & & 0 & 0 & \cdots & 0 & 0 & 1 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}, \quad N^{n-2} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ & & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}, \quad N^{n-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

so the series expression of e^{Nt} will terminate after n terms,

$$e^{Nt} = I + tN + \frac{1}{2}(tN)^2 + \cdots + \frac{1}{(n-1)!}(tN)^{n-1} + \mathbf{0}$$

this gives us

$$e^{Jt} = e^{\lambda t}\begin{bmatrix} 1 & t & t^2/2 & t^3/6 & \cdots & t^{n-3}/(n-3)! & t^{n-2}/(n-2)! & t^{n-1}/(n-1)! \\ & 1 & t & t^2/2 & \cdots & t^{n-4}/(n-4)! & t^{n-3}/(n-3)! & t^{n-2}/(n-2)! \\ & & 1 & t & \cdots & t^{n-5}/(n-5)! & t^{n-4}/(n-4)! & t^{n-3}/(n-3)! \\ & & & 1 & \cdots & t^{n-6}/(n-6)! & t^{n-5}/(n-5)! & t^{n-4}/(n-4)! \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 1 & t & t^2/2 \\ & & & & & & 1 & t \\ & & & & & & & 1 \end{bmatrix}$$

Let us consider a small example.

Example 1.1.3

a) Find e^{At} when

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

$$A = -1I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

so we have

$$e^{At} = e^{-t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

b) Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We know that the solution is given by

$$\mathbf{y} = e^{At}\mathbf{y}(0) = e^{-t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}\begin{bmatrix} -1 \\ 1 \end{bmatrix} = e^{-t}\begin{bmatrix} -1+t \\ 1 \end{bmatrix}$$

Example 1.1.4

Let A be the matrix

$$\begin{bmatrix} -7 & 9 \\ -4 & 5 \end{bmatrix}.$$

The characteristic polynomial is $\rho_A(\lambda) = (\lambda+1)^2$, so A has the eigenvalue $\lambda = -1$

with algebraic multiplicity 2. The eigenspace is spanned by $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We can't find another linearly independent eigenvector of A, but we can find a **generalized eigenvector** $\mathbf{w} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$ which satisfies

$$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

The matrix $P = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} 3 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}$ with columns $\mathbf{v}$ and $\mathbf{w}$ is invertable with inverse

$$P^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -2 & 3 \end{bmatrix}.$$

From this we have

$$A = PJP^{-1}$$

where J is the matrix $J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ as in the last example.

The matrix exponential e^{At} is therefore given as

$$e^{At} = Pe^{Jt}P^{-1}$$

Now, if we would like to have the general solution of the system $\mathbf{y}' = A\mathbf{y}$, then we can do as for the diagonalizable case above, namely a variable change. Set $\mathbf{x}(t) = P{-}1\mathbf{y}(t)$, then we have that the general solution of

$$\mathbf{x}' = J\mathbf{x}$$

is equal to

$$\mathbf{x}(t) = e^{Jt}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} t \\ 1 \end{bmatrix}$$

The general solution to $\mathbf{y}' = A\mathbf{y}$ is therefore

$$\begin{aligned} \mathbf{y}(t) = P\mathbf{x}(t) &= \begin{bmatrix} 3 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}\left(c_1 e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} t \\ 1 \end{bmatrix}\right) \\ &= c_1 e^{-t}\begin{bmatrix} 3 \\ 3 \end{bmatrix} + c_2 e^{-t}\begin{bmatrix} 3t - 1/2 \\ 2t \end{bmatrix} \\ &= c_1 e^{-t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 e^{-t}\left(t\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}\right). \end{aligned}$$

From this example we obtain a procedure for finding the general solution of

2×2-systems with only one eigenvalue.

Theorem 1.1.2

Let A be a real 2×2-matrix with a real eigenvalue λ of algebraic multiplicity 2. Let $\mathbf{v}$ be an eigenvector of λ, and let $\mathbf{w}$ be a vector solving $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then the solutions of the system $\mathbf{y}' = A\mathbf{y}$ are on the form

$$\mathbf{y}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}(t\mathbf{v} + \mathbf{w}).$$

Complex eigenvalues

If A is a 2×2-matrix with complex eigenvalues $a \pm ib$ and real coefficients, we already know that the general solution of $\mathbf{y}' = A\mathbf{y}$ is given by

$$\mathbf{y}(t) = c_1 e^{a+ib}\mathbf{v}_1 + c_2 e^{a-ib}\mathbf{v}_2$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors belonging to the eigenvalues. However, these solutions are in general complex-valued. Hence, we can ask ourselves whether we can find a general expression for the real-valued solutions.

We start by stating a result without proving or arguing for it.

Theorem 1.1.3

Let A be a real 2×2-matrix with a complex eigenvalue $\lambda = a - bi$, $b \neq 0$, and let $\mathbf{v} = \begin{bmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \end{bmatrix} \in \mathbb{C}^2$ be an eigenvector belonging to λ. Then we can factor A as following:

$$A = PCP^{-1} \text{ with } P = \begin{bmatrix} \mathrm{Re}\mathbf{v} & \mathrm{Im}\mathbf{v} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$$

and

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

The form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is particularly nice, since we can rewrite it as a sum of commuting matrices

$$C = aI_2 + bS, \quad \text{for } S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The commutativity of them gives us that

$$e^{Ct} = e^{aI}e^{bSt} = e^{at}e^{bSt}$$

Let us try to find the last matrix exponential. We first observe that S^i is periodic

$$S^0 = I_2, \quad S^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad S^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2, \quad S^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -S$$

$$S^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \quad S^5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = S, \quad S^6 = -I_2 = S^2, \quad S^7 = -S = S^3, \quad \dots$$

In the power series of e^{bSt} we can group together the even and odd powers of S, and get

$$\begin{aligned} e^{bSt} = &\left(1 - \frac{1}{2!}(bt)^2 + \frac{1}{4!}(bt)^4 - \frac{1}{6!}(bt)^6 + \frac{1}{8!}(bt)^8 - \cdots\right) I_2 \\ &+ \left(bt - \frac{1}{3!}(bt)^3 + \frac{1}{5!} - \frac{1}{7!}(bt)^7 + \frac{1}{9!}(bt)^9 - \cdots\right) S \end{aligned}$$

Recognize that the first part is equal to the taylor series of $\cos(bt)$

$$\cos(bt) = 1 - \frac{1}{2!}(bt)^2 + \frac{1}{4!}(bt)^4 - \frac{1}{6!}(bt)^6 + \frac{1}{8!}(bt)^8 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

and the second part is equal to the taylor series of $\sin(bt)$

$$\sin(bt) = bt - \frac{1}{3!}(bt)^3 + \frac{1}{5!} - \frac{1}{7!}(bt)^7 + \frac{1}{9!}(bt)^9 - \cdots = \sum_{n=0}^{\infty}(-1)^{n-1} \frac{x^{2n+1}}{(2n+1)!}.$$

Hence, we see that

$$e^{bSt} = \cos(bt)I_2 + \sin(bt)S = \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}$$

Now, using the substitution $\mathbf{x} = P^{-1}\mathbf{y}$ in $\mathbf{y}' = A\mathbf{y}$, with P as in the above theorem, we get

$$\mathbf{x}(t) = c_1 e^{at} \begin{bmatrix} \cos(bt) \\ \sin bt \end{bmatrix} + c_2 e^{at} \begin{bmatrix} -\sin(bt) \\ \cos(bt) \end{bmatrix}$$

and using $\mathbf{y}(t) = P\mathbf{x}(t)$

$$\begin{aligned} \mathbf{y}(t) = &\begin{bmatrix} \mathrm{Re}\mathbf{v} & \mathrm{Im}\mathbf{v} \end{bmatrix} \left(c_1 e^{at} \begin{bmatrix} \cos(bt) \\ \sin bt \end{bmatrix} + c_2 e^{at} \begin{bmatrix} -\sin(bt) \\ \cos(bt) \end{bmatrix} \right) \\ = &c_1 e^{at}\cos(bt)\mathrm{Re}(\mathbf{v}) + c_1 e^{a} t \sin(bt)\mathrm{Im}(\mathbf{v}) \\ &- c_2 e^{at}\sin(bt)\mathrm{Re}(\mathbf{v}) + c_2 e^{at}\cos(bt)\mathrm{Im}(\mathbf{v}) \\ = &c_1 e^{at}[\cos(bt)\mathrm{Re}(\mathbf{v}) + \sin(bt)\mathrm{Im}(\mathbf{v})] \\ &+ c_2 e^{at}[\cos(bt)\mathrm{Im}(\mathbf{v}) - \sin(bt)\mathrm{Re}(\mathbf{v})] \end{aligned}$$

Theorem 1.1.4

Let A be a 2×2-matrix with a complex eigenvalue $a - ib$ and real entries. Let also $\mathbf{v}$ be an eigenvector of $a - ib$, then the general solution of the system

$$\mathbf{y}' = A\mathbf{y}$$

is given by

$$\begin{aligned}\mathbf{y}(t) =& c_1 e^{at}[\cos(bt)\mathrm{Re}(\mathbf{v}) + \sin(bt)\mathrm{Im}(\mathbf{v})] \\ &+ c_2 e^{at}[\cos(bt)\mathrm{Im}(\mathbf{v}) - \sin(bt)\mathrm{Re}(\mathbf{v})]\end{aligned}$$

Example 1.1.5

Let us find the solutions of the system $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues are $\pm i$. You can verify that $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector belonging to i. We find

$$\mathrm{Re}\mathbf{v} = \mathrm{Re}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{Re} i \\ \mathrm{Re} 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\mathrm{Im}\mathbf{v} = \mathrm{Im}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{Im} i \\ \mathrm{Im} 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Theorem 1.1.4 tells us that the general solution is spanned by

$$\begin{aligned}& e^{\alpha t}(\mathrm{Re}(\mathbf{v})\cos(\beta t) - \mathrm{Im}(\mathbf{v})\sin(\beta t)) \\ =& e^{0 \cdot t}(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(1 \cdot t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(1 \cdot t)) \\ =& \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned}
&e^{\alpha t}(\mathrm{Re}(\mathbf{v})\sin(\beta t) + \mathrm{Im}(\mathbf{v})\cos(\beta t)) \\
=&e^{0\cdot t}(\begin{bmatrix}0\\1\end{bmatrix}\sin(1\cdot t) + \begin{bmatrix}1\\0\end{bmatrix}\cos(1\cdot t)) \\
=&\begin{bmatrix}\cos t\\ \sin t\end{bmatrix}.
\end{aligned}$$

So the general solution is

$$\begin{aligned}
&c_1\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} + c_2\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix} \\
=&\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}.
\end{aligned}$$

We recognize this as

$$\mathbf{y}(t) = e^{At}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

Example 1.1.6

Let

$$A = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}.$$

The complex eigenvalues are $1 \pm i$. Verify that $\begin{bmatrix}i\\1\end{bmatrix}$ is an eigenvector. Theorem 1.1.4 tell us that the general solution is spanned by

$$e^t\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} \quad \text{and} \quad e^t\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}.$$

The general solution is

$$e^t\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}c_1\\c_2\end{bmatrix}.$$

which we recognize as

$$\mathbf{y}(t) = e^{At}\begin{bmatrix}c_1\\c_2\end{bmatrix}$$

1.2 From nth order to system

The observant student may already have seen that the three cases above seem quite similar to the solution cases for a second order ODE. This is not just a quirky coincidence, but a consequence of the following relations.

Say that we have a third order linear ODE with constant coefficients

$$y''' + a_2 y'' + a_1 y' + a_0 y = 0$$

In order to solve this we could try to introduce a few more functions, say

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \\ x_3(t) &= y''(t) \end{aligned}$$

then we can rewrite our ODE as a system of first order ODEs

$$\begin{array}{rcrrr} x_1' & = & & x_2 & \\ x_2' & = & & & x_3 \\ x_3' & = & -a_0 x_1 & -a_1 x_2 & -a_2 x_3 \end{array}$$

or in matrix notation

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In general we have

Theorem 1.2.1 Converting to system

An nth-order ODE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(t)$$

can be converted to a system of first order ODEs

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_{n-1}' \\ y_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$$

where

$$\begin{aligned} y_1 &= y \\ y_2 &= y' \\ y_3 &= y'' \\ &\vdots \\ y_{n-1} &= y^{(n-2)} \\ y_n &= y^{(n-1)} \end{aligned}$$

Second order ODEs

Now, let us consider the homogeneous second order ODEs,

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

Rewriting this as a system, we get

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \mathbf{y}$$

We have seen how we can solve these systems above, we start by finding the characteristic polynomial

$$\rho(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -a_0 & -a_1 - \lambda \end{vmatrix} = \lambda(\lambda + a_1) - (-a_0) = \lambda^2 + \lambda a_1 + a_0$$

and then depending on whether this polynomial has real, complex or a single root we find eigenvector(s) (and generalized eigenvectors). We can now see that this polynomial is the same that we considered when solving the second order ODEs earlier. Let us venture on further, we claim that if λ is a root of $\rho(\lambda)$ (i.e. an eigenvalue), then $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$ is an eigenvector of the matrix:

$$\begin{aligned} \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda \end{bmatrix} &= \begin{bmatrix} \lambda \\ -a_0 - a_1\lambda \end{bmatrix} \\ &= \begin{bmatrix} \lambda \\ \lambda^2 \end{bmatrix} \text{ since } \lambda \text{ is a root of } \rho(\lambda) \\ &= \lambda \begin{bmatrix} 1 \\ \lambda \end{bmatrix}. \end{aligned}$$

Two real eigenvalues

Now, if

$$\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$$

has two real eigenvalues, we know that the system is solved by

$$\mathbf{y} = c_1 e^{\lambda t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{bmatrix}$$

and if we read out the first entry of the solutions we rediscover the solution $y(t)$ of the ODE:

$$y(t) = y_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Only one eigenvalue

The only way for us to get exactly one eigenvalue is if the expression under the root sign in

$$\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

is zero. That is $a_1^2 = 4a_0$ and $\lambda = \frac{-a_1}{2}$.

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -4a_0 \end{bmatrix}$$

In order to find a solution of the system we therefore need to find a generalized eigenvector, that is solve

$$\left(\begin{bmatrix} 0 & 1 \\ -a_0 & -4a_0 \end{bmatrix} - \begin{bmatrix} -\frac{a_0}{2} & 0 \\ 0 & -\frac{a_0}{2} \end{bmatrix} \right) \mathbf{w} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

The left hand side is equal to

$$\begin{bmatrix} \frac{a_1}{2} & 1 \\ -a_0 & -\frac{a_0}{2} \end{bmatrix} \mathbf{w}$$

and we see that $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a solution. Thus our general solution of the system is

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{\lambda t} \mathbf{v} + c_2 e^{\lambda t} (t\mathbf{v} + \mathbf{w}) \\ &= c_1 e^{-a_0/2t} \begin{bmatrix} 1 \\ -a_0/2 \end{bmatrix} + c_2 e^{-a_0/2} \left(t \begin{bmatrix} 1 \\ -a_0/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} c_1 e^{-a_0/2t} + c_2 t e^{-a_0/2t} \\ c_1 e^{a_0 t}(1 - a_0/2) - c_2 t e^{-a_0 t} \end{bmatrix} \end{aligned}$$

The solution of the ODE is now read out from the first entry,

$$y(t) = y_1(t) = c_1 e^{-a_0/2t} + c_2 t e^{-a_0/2t} = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

which is the same solution as we had earlier.

Two complex eigenvalues

If we have comples eigenvalues $\lambda_\pm = a \pm ib$, then from Theorem 1.1.4 the solution of the system is given by

$$\begin{aligned}\mathbf{y}(t) =& c_1 e^{at}[\cos(bt)\mathrm{Re}(\mathbf{v}) + \sin(bt)\mathrm{Im}(\mathbf{v})] \\ &+ c_2 e^{at}[\cos(bt)\mathrm{Im}(\mathbf{v}) - \sin(bt)\mathrm{Re}(\mathbf{v})]\end{aligned}$$

Further, since we know that an eigenvector is given by

$$\mathbf{v} = \begin{bmatrix} 1 \\ \lambda_- \end{bmatrix} = \begin{bmatrix} 1 \\ a - ib \end{bmatrix},$$

we have

$$\mathrm{Re}\mathbf{v} = \begin{bmatrix} 1 \\ a \end{bmatrix} \quad \text{and} \quad \mathrm{Im}\mathbf{v} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

and

$$\begin{aligned}\mathbf{y}(t) =& c_1 e^{at}\left[\cos(bt)\begin{bmatrix} 1 \\ a \end{bmatrix} + \sin(bt)\begin{bmatrix} 0 \\ b \end{bmatrix}\right] \\ &+ c_2 e^{at}\left[\cos(bt)\begin{bmatrix} 0 \\ b \end{bmatrix} - \sin(bt)\begin{bmatrix} 1 \\ a \end{bmatrix}\right]\end{aligned}$$

If we read out the first entry of this solution we get the solution of the ODE

$$y(t) = y_1(t) = e^{at}[c_1 \cos(bt) + c_2 \sin bt]$$

which also is the one we obtained earlier.

1.3 Phase portrait

As when we worked with ODEs in the start, we can obtain some information about the solutions of a system of ODEs through a graphical representation. Specifically the systems consisting of two ODEs.

We know that the system

$$\mathbf{y}' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \mathbf{y}$$

has a unique solution for every initial condition $\mathbf{y}(0)$. These solutions can be drawn as trajectories in the $y_1 y_2$-plane.

Example 1.3.1

Consider the system

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}$$

which has general solution

$$\mathbf{y}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

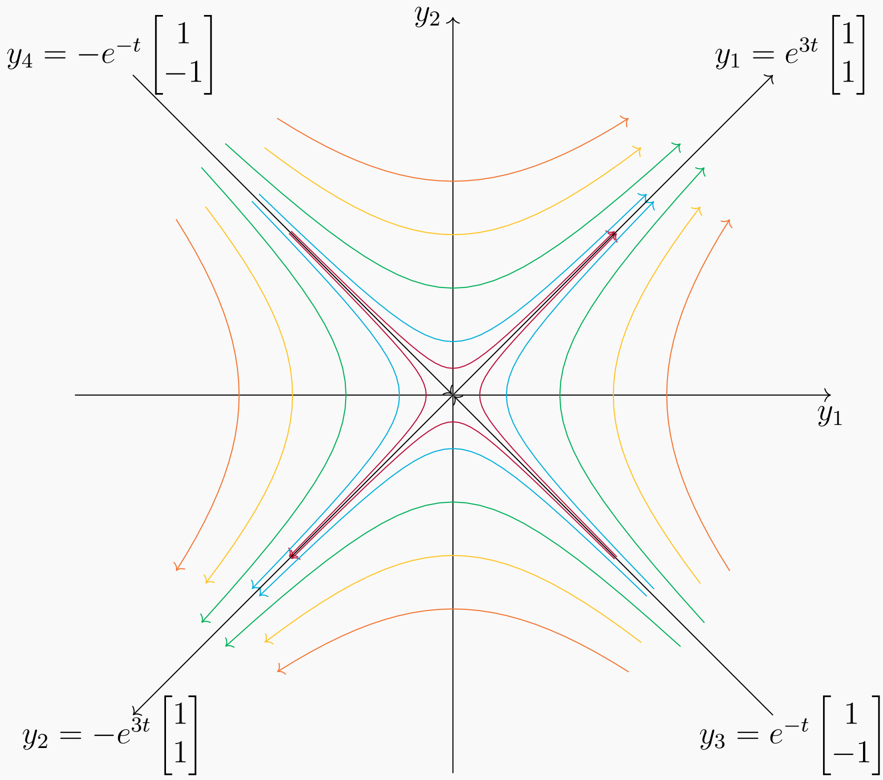
Below we have drawn different solutions of this system. Note specifically the black solution curves that correspond to

$$y_1(t) = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_2(t) = -e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_3(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y_4(t) = -e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

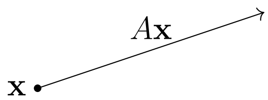


Vector field

To each 2×2 system

$$\mathbf{y}' = A\mathbf{y}$$

we have an associated vector field. For each point $\mathbf{x}$ in the $\mathbf{y}_1\mathbf{y}_2$-plane we can associate a vector $A\mathbf{x}$ starting in $\mathbf{x}$.

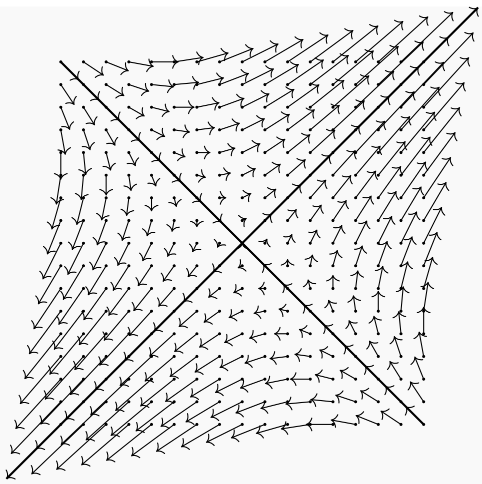


$A\mathbf{x}$ as a vector starting in $\mathbf{x}$

Example 1.3.2

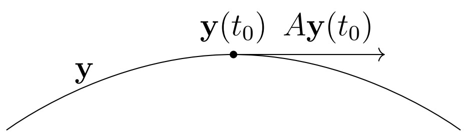
Here is a sketch of the vector field associated to the system with

$$A = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}.$$



The axes in bold are the lines spanned by the eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ og $\begin{bmatrix}1\\-1\end{bmatrix}$ which correspond to the eigenvectors 3 og -1. Do notice how the arrows move towards infinity along $\begin{bmatrix}1\\1\end{bmatrix}$ which correspond to a positive eigenvalue; towards the origin along $\begin{bmatrix}1\\-1\end{bmatrix}$ which correspond to a negative eigenvalue.

How does the vector field help us to understand the solution of systems? A solution $\mathbf{y}$ of $\mathbf{y}' = A\mathbf{y}$ is a curve which satisfies the derivative in a point of time t_0, $\mathbf{y}'(t_0) = A\mathbf{y}(t_0)$. The derivative is in other words, the vector in $\mathbf{y}(t_0)$ from the vector field associated to A.



The vector $A\mathbf{y}(t_0)$ is the derivative of $\mathbf{y}$ in $t = t_0$

The arrows lie tangent to the solution curves.

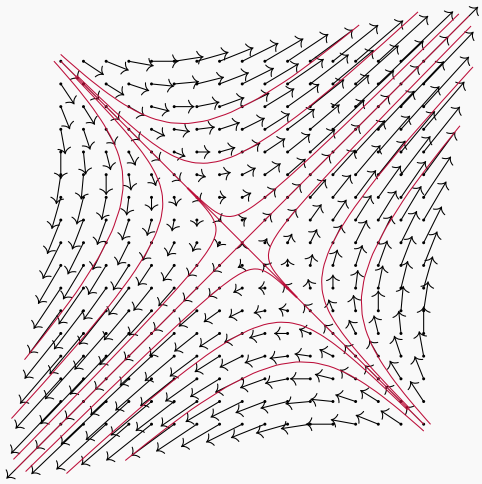
A phase portrait of $\mathbf{y}' = A\mathbf{y}$ is a sketch of all possible solutions, including the the orientation, i.e. which direction we move along the curves for increasing t. One way of making a phase portrait is to first make a sketch of the vector field of A and then drawing curves along the arrows.

Example 1.3.3

Let us look back at the system with

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

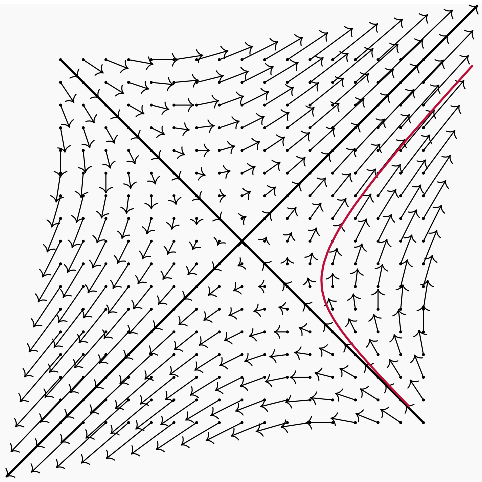
Draw some curves that have the arrows in the vector field of A as tangents to obtain a phase portrait:



The solutions move towards the origin along $\operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$, and then bends before moving away from the origin along $\operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. The reason for this behavior is that the term $c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$ dominates for negative t (negative eigenvalue), and $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$ dominates for positive t (positive eigenvalue).

Example 1.3.4

When we look at a solution that moves through a given point $\mathbf{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, we have a unique curve in the vector field:



Let us now discuss in more detail two of the different cases of solutions we have for a two dimensional system, namely

1. two different real eigenvalues, and

2. two complex eigenvalues.

Two real eigenvalues

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two linearly independent eigenvectors belonging to the eigenvalues λ_1 og λ_2. Then we know that

$$c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}$$

is a general solution of $\mathbf{y}' = A\mathbf{y}$.

We can classify all possible phase portraits based on the eigenvalues and eigenvectors. The factor $e^{\lambda t}$ tells us how the solutions moves along the span of $\mathbf{v}$ when t changes:

λ	$e^{\lambda t}$	$\mathbf{v}e^{\lambda t}$
> 0	increases	away from the origin
$= 0$	constant	don't move
< 0	decreases	towards the origin

Table 1.1: What happens when t grows?

Be aware that $\lambda < 0$ dominates when $t \ll 0$; $\lambda > 0$ dominates when $t \gg 0$. For a given system we have two such terms in the solution Here is a method to sketch the phase portrait of $\mathbf{y}' = A\mathbf{y}$ when A have two real eigenvalues:

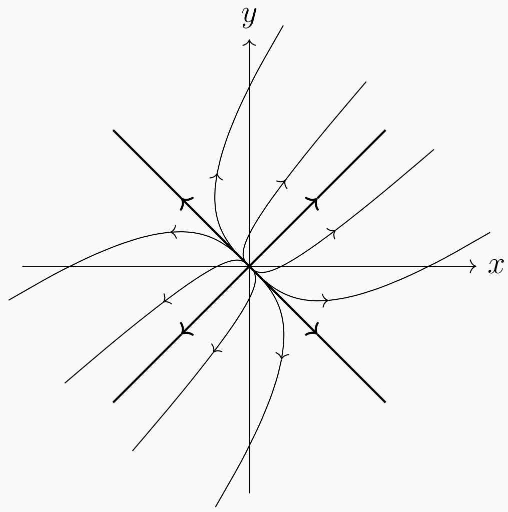
(a) Draw the span of two linearly independent eigenvectors in the plane.
(b) Determine the movement of the solutions along the span of each eigenvector.
(c) Draw curves that move according to point b)

Example 1.3.5

The system with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

has two linearly independent eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalues 3 and 1. From the discussion above, we know that the solutions move away from the origin along both eigenvectors. Notice that $e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ dominates for big t. The solutions between the axis spanned by the eigenvectors will therefore become increasingly parallell to $\pm \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ when t grows larger.



Two complex eigenvalues

When we have complex eigenvalues we know that the solution is given by

$$\begin{aligned} \mathbf{y}(t) =& c_1 e^{at}[\cos(bt)\mathrm{Re}(\mathbf{v}) + \sin(bt)\mathrm{Im}(\mathbf{v})] \\ &+ c_2 e^{at}[\cos(bt)\mathrm{Im}(\mathbf{v}) - \sin(bt)\mathrm{Re}(\mathbf{v})] \end{aligned}$$

The terms with cosine and sine gives a circular movement. If $\alpha \neq 0$ we also have in addition an inward or outward motion, dependent on the sign of α –as in the real case. The combination of these two motion are spirals which either moves inward to the origin or outward from the origin. Vi oppsummerer:

α	movement
> 0	outward going spirals
$= 0$	circular
< 0	inward moving spirals

Example 1.3.6

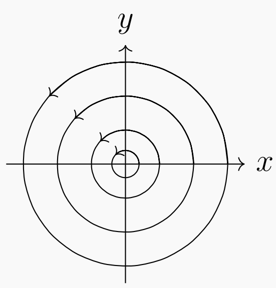
Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

be as in the example (1.1.5), where we found the general solution

$$\begin{aligned} & c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \\ = & \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \end{aligned}$$

The phase portrait consists of circles centered in the origin, oriented anti-clockwise.



Example 1.3.7

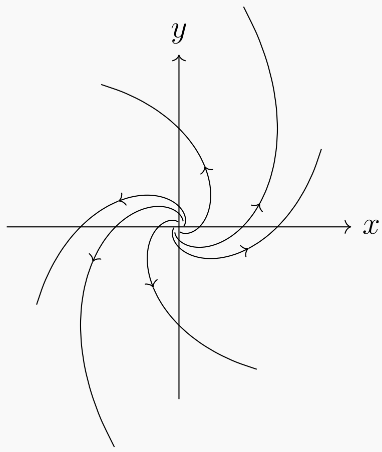
Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

be as in Example (1.1.6). We found the solution

$$e^t \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The factor e^t contributes to an outward motion, while the matrix gives a circular motion in an anti-clockwise direction. The phase portrait consists of outward moving spirals oriented anti-clockwise.



We knew that the motion was anti-clockwise since we recognized the rotation matrix. A more methodical way would be to plot the vectorfield of A in a point or two. Often $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ or $\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ are convenient points for these, since they are easily plotted. For example

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

so the vector field is sloped upwards in the first quadrant from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Thus the spirals, which are tangent to the field, moves anti-clockwise.