An introductory Course to Linear Algebra and Differential Equations

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Based on notes from the courses TMA4110/4115

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Lecture notes 2022

Contents

Part I

Differential equations

0.1 What is a differential equation?

A **differential equation** is an equation describing unknown functions through their derivatives. For example

$$\frac{d}{dx}y = 2x$$

is a differential equation where the unknown function y is of the form $y = x^2 + C$ for some C. A slightly more sophisticated example is the equation

$$y'(x) = y(x)$$

which can be solved by the function e^x. The land of differential equations is vast and most of it hard to traverse. We will in this course take a scenic route through the least rough parts of the theory and have a slight look at how modern technology will come to our aid outside of this path.

The first step will be to tell you that we will be looking at only **ordinary differential equations**, ODE for short. This means that all derivatives will be ordinary, i.e.

$$\frac{d}{dx}y(x),\ \frac{d^2}{dx^2}y(x),\ \ldots,\ \frac{d^n}{dx^n}y(x)$$

and not partial, i.e.

$$\frac{\partial}{\partial x}u(x,t),\ \frac{\partial}{\partial t}u(x,t),$$
$$\frac{\partial^2}{\partial x^2}u(x,t),\ \frac{\partial^2}{\partial t^2}u(x,t),\ \frac{\partial^2}{\partial x\partial t}u(x,t),\ \frac{\partial^2}{\partial t\partial x}u(x,t)$$

Equivalently, the unknown functions in this course will only be dependent on one variable. If the variable is time, we will sometimes be using the dot notation, $\dot{y}$, to denote the derivative of y.

Before moving we introduce two extra descriptive words which we will be using. The **order** of a ODE denotes the largest degree of differentiation occurring in the equation. For example $\dot{y} = ay$ which gives us exponential growth for $a > 0$ and exponential decay for $a < 0$, is a first order differential equation. The ODE $mx'' = -kx + mg$ is a second order differential equation, describing a vibrating string.

The notion of **linear** ODE's tells us that the unknown function is given by linear expressions in the equation. That is, $\dot{y} = ay$ is linear, but $\dot{y} = y^2$ is not. In it's most general form a linear ODE looks like

$$y^{(n)} = a_1(t)y^{(n-1)} + a_2(t)x^{(n-2)} + \cdots + a_{n-1}(t)y' + a_n(t)y + g(t)$$

where $y^{(i)}$ is the ith derivative of y and $a_i(t)$ are functions over t.

Solutions

For a function $y(t)$ to be a **solution** of a differential equation, it has to satisfy the equation for every t in some interval I. This interval may be all of $\mathbb{R}$. For time-dependent systems it might also be natural to restrict to the interval of non-negative numbers $[0, \infty)$.

Examples

Differential equations has a broad impact on modern science. They appear frequently in mathematical models attempting to describe real-life phenomenon.

Example 0.1.1 Newton's second law of motion

Newton's second law of motion states that:

> When a body is acted upon by a force, the time rate of change of its momentum equals the force.

Written as a differential equation, the same law states

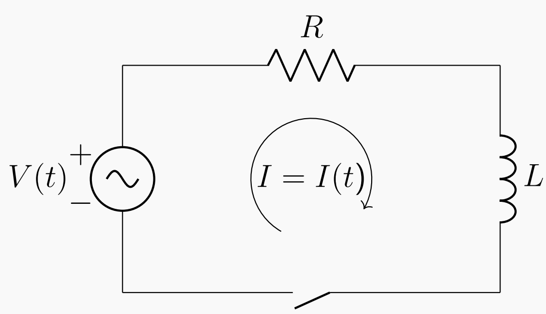
$$m\frac{d}{dt}v = F$$

where m is the mass of the object, v is the speed and F is the acting force. In this equation the unknown function may be the speed of the object v.

More involved differential equations may be found when working on electrical circuits.

Example 0.1.2

Let us look at the RL-circuit



which has a resistance R, an inductance L, and a generator that supplies a voltage

$V(t)$ when the switch is closed. The current $I = I(t)$ in the circuit satisfies the linear first-order ODE

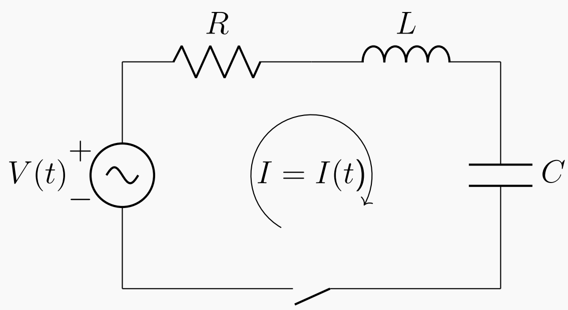
$$L\frac{dI}{dt} + RI = V(t)$$

The general solution for I is given by

$$I(t) = e^{-(R/L)t}\left[c + \frac{1}{L}\int V(t)e^{(R/L)t}dt\right]$$

Example 0.1.3

In the RLC-circuit



The charge $Q = Q(t)$ in the capacitor satisfies the second-order linear non-homogeneous ODE

$$L\frac{d^2Q}{dt^2} + R\frac{dQ}{dt} + \frac{1}{C}Q = V(t)$$

Chapter 1

First order equations

1.1 First order linear ODE's

We start our journey with one of the simplest kind of differential equations.

Definition 1.1.1

A first-order linear ODE is on the form

$$\frac{dy}{dt} + f(t)y = g(t)$$

where $f(t)$ and $g(t)$ are known functions. Or, written with the dot-notation

$$\dot{y} + f(t)y = g(t)$$

These kind of equations have a straight-forward solutions formula, using what we call the **integrating factor**:

$$e^{F(t)}, \text{ for } F(t) = \int f(t)dt$$

Theorem 1.1.1 Solving first-order linear ODE's

When solving

$$\frac{dy}{dt} + f(t)y = g(t)$$

for f and g continuous on a interval I, you

1. Calculate
$$e^{F(t)} = e^{\int f(t)dt}$$
2. Multiply $e^{F(t)}$ with the equation. The equation now looks like
$$e^{F(t)}\dot{y}(t) + e^{F(t)}f(t)y(t) = e^{F(t)}g(t),$$
3. The left hand side can now be recognized as the product derivative
$$\frac{d}{dt}\left(e^{F(t)}y(t)\right),$$
so the equation is now
$$\frac{d}{dt}\left(e^{F(t)}y(t)\right) = e^{F(t)}g(t).$$
4. Integrating both sides gives
$$e^{F(t)}y(t) = \int e^{F(t)}g(t)\ dt + C,$$
5. Solving with respect to $y(t)$ now gives us the general solution
$$\begin{aligned} y(t) &= e^{-F(t)}\left(\int e^{F(t)}g(t)\ dt + C\right) \\ &= e^{-F(t)}\int e^{F(t)}g(t)\ dt + e^{-F(t)}C \end{aligned} \tag{1.1}$$

It is best to learn the method, rather than to try and remember formula 1.1.

Example 1.1.1

Let us illustrate the method by an example. We solve the following ODE
$$\dot{y} - \frac{1}{4}y = \frac{1}{8}$$
which is on the form $\dot{y} + f(t)y = g(t)$ with $f(t) = -\frac{1}{4}$ and $g(t) = \frac{1}{8}$. Let us start by calculation the integrating factor:
$$e^{F(t)} = e^{\int -1/4dt} = e^{-1/4t}.$$

Observe that we set $C = 0$ in the undetermined integral $\int f(t)dt$. We only need an antiderivative of f in the exponent, so we can do this in general. We multiply the equation with our factor and get

$$\begin{aligned} e^{-1/4t}\dot{y} - e^{-1/4t}\frac{1}{4}y &= e^{-1/4t}\frac{1}{8} \\ \frac{d}{dt}\left(e^{-1/4t}y\right) &= \frac{1}{8}e^{-1/4t} \\ e^{-1/4t}y &= \int \frac{1}{8}e^{-1/4t}dt \\ e^{-1/4t}y &= -\frac{1}{2}e^{-1/4t} + C \\ y(t) &= -\frac{1}{2} + Ce^{1/4t} \end{aligned}$$

Observe that we have a solution for every value of C.

Example 1.1.2

We solve the equation

$$y' = ay$$

where $a > 0$.

This can be written as

$$y' - ay = 0,$$

and we multiply this by e^{-at}, giving

$$e^{-at}y'(t) - e^{-at}ay(t) = 0,$$

and thus

$$\frac{d}{dt}\left(e^{-at}y(t)\right) = 0.$$

After integration, we have

$$e^{-at}y(t) = C,$$

or rather

$$y(t) = Ce^{at},$$

Example 1.1.3

We now consider the problem

$$y'(t) - \cos(t)y(t) = e^{\sin t}.$$

We multiply by

$$e^{-\int \cos t \; dt} = e^{-\sin t},$$

giving

$$\frac{d}{dt}\left(e^{-\sin t}y(t)\right) = 1.$$

After integration, we get

$$e^{-\sin t}y(t) = t + C,$$

os the answer is

$$y(t) = te^{\sin t} + Ce^{\sin t}.$$

Often we know what the unknown function evaluates to for a particular point in time. For example if we are looking for the current $I = I(t)$ in an electrical curcuit, we might assume that at $t = 0$ there is no current at all, i.e. $I(0) = 0$. If we have such constraints on the solution we have an **initial value problem**, or IVP for short.

Example 1.1.4

Find the general solution of

$$\dot{y} + \frac{1}{t} \cdot y = t^2 + 1$$

for $t > 0$. Find also the solution to the initial value problem we get when also assuming $y(1) = 1$.

Integrating factor:

$$e^{F(t)} = e^{\int \frac{1}{t} dt} = e^{\ln t} = t$$

Multiplying t and the equation gives us:

$$\begin{aligned} t\dot{y} + y &= t^3 + t \\ \frac{d}{dt}(ty(t)) &= t^3 + t \\ ty(t) &= \int (t^3 + t)dt \\ y(t) &= \frac{1}{4}t^3 + \frac{1}{2}t + \frac{1}{t}C \end{aligned}$$

Which is the general solution. Now, if $y(1) = 1$ we see that

$$1 = \frac{1}{4} \cdot 1^3 + \frac{1}{2} \cdot 1 + C \implies C = \frac{1}{4}$$

The particular solution for the initial value problem is thus

$$y(t) = \frac{1}{4}t^3 + \frac{1}{2}t + \frac{1}{4t} = \frac{1}{4}(t^3 + 2t + 1)$$

1.2 Existence and uniqueness

In the last example we saw that when adding an extra constraint on the ODE we got a unique solution. This holds in general and we put it in a nice little box to commemorate it.

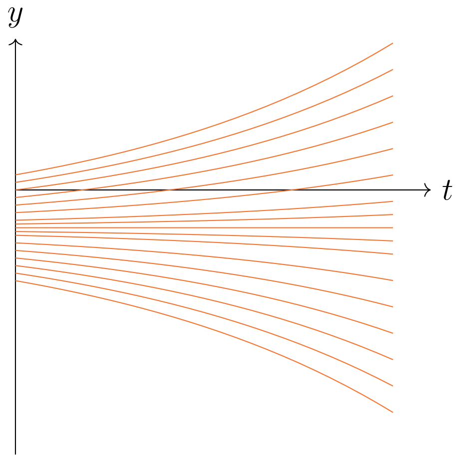
Theorem 1.2.1

Assume that f and g are continuous functions on the open interval I. Let t_0 be any number in I and assume that y_0 is any number in $\mathbb{R}$. Then there exists exactly one solution of the differential equation

$$\frac{dy}{dt} + f(t)y = g(t), \quad t \in I$$

such that $y(t_0) = y_0$.

This result tells us that two solutions of a first-order, linear ODE never intersects. So, if we plot every solution, we obtain a collection of curves that "sits beside eachother". Here we give such a plot for a collection of solutions to the ODE of example 1.1.1:



Solution curves of $\dot{y} - \frac{1}{4}y = \frac{1}{8}$

This is an example of a **direction field**, which may be useful to get a geometric intuition on how the solutions of a ODE behaves.

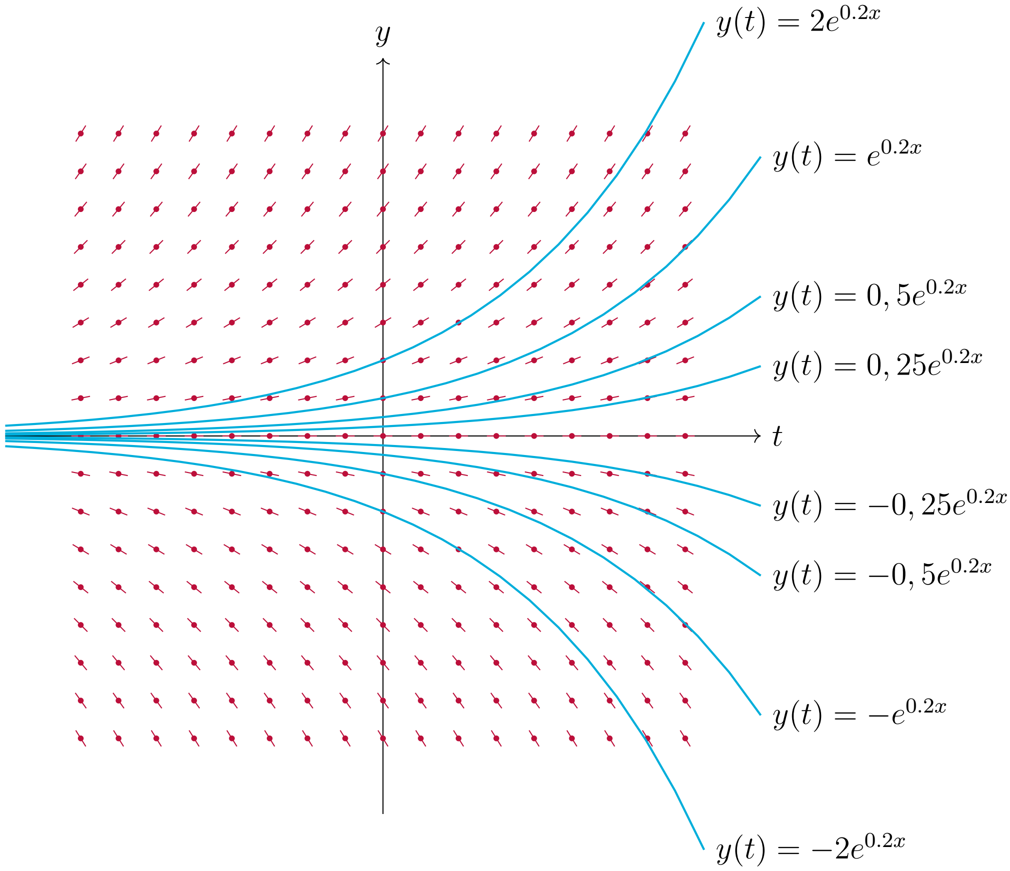
1.3 Direction field

Let us illustrate direction fields through another example. Consider the ODE

$$\dot{y} = ay$$

Let $y = y(t)$ be a solution to this equation. If (t_0, y_0) is a point on the graph of the solution, then we can know from the differential equation that the slope of the solution curve in (t_0, y_0) is given by $\dot{y_0} = ay_0$. If we draw a short line segment in the given point with this slope, we know that the solution cruve has this segment as a tangent. Now, if we draw a such line segments for a grid of points, we can get an idea of how the solutions behave.

Since each line segment is a tangent of a solution, we can use them to sketch possible solution curves. For each point, we know there will only correspond one possible solution.



Direction field of $\dot{y} = 0.2y$ with different solution curves sketched.

Example 1.3.1

Let us sketch the direction field of

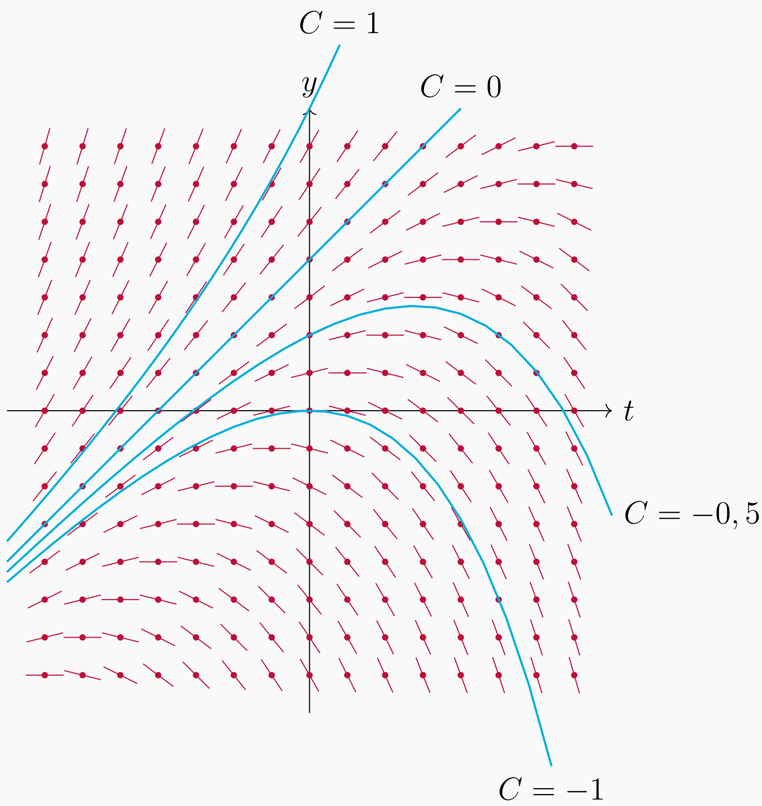
$$\dot{y} = y - t$$

and some solution curves of it.

The general solution of the ODE is

$$y(t) = Ce^t + t + 1$$

and the directional plot with some solution curves is



Equilibrium points

Can we determine the behaviour of the ODE

$$\dot{x}(t) = f(x(t)),$$

without solving the problem explicitly? How does the solution depend on the initial data?

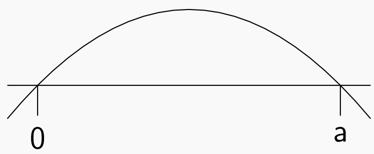
Definition 1.3.1 Equilibrium Point

If $f(t_0) = 0$, t_0 is an **equilibrium point**. These are points at which $\dot{x}(t_0) = 0$, so $x(t)$ stays at x_0 for all t.

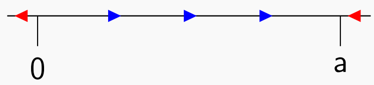
Example 1.3.2

Consider $f(x) = -x(x-a)$, for some $a \in \mathbb{R}$. Then $f(x) = 0$ for $x = 0$ and $x = a$. Thus both of these are equilibrium points. Assume $a > 0$.

What about if we are outside of these equilibrium points? We wish to examine the behaviour in these situations.



Plot of f from Example 1.3.2.

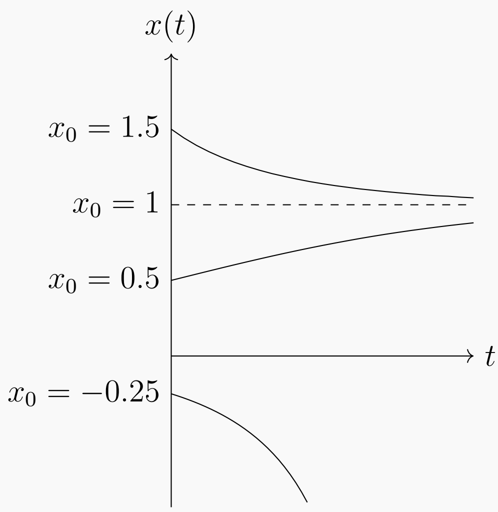


Plot of increasing and decreasing regions
for the solution given some initial data, from Example 1.3.2.

Upon plotting f, we see that there are 3 regions, in which there 2 different behaviours are exhibited. For a choice of $x_0 \in \mathbb{R}$ we have for t near 0 that

- $x_0 < 0$: Then $\dot{x}(t) \approx f(x_0) < 0$, so we are moving to the left,
- $0 < x_0 < a$: Then $\dot{x}(t) \approx f(x_0) > 0$, so we are moving to the right,
- $a < x_0$: Then $\dot{x}(t) \approx f(x_0) < 0$, so we are moving to the left,

These regions are plotted here



We name the equilibrium points based on the behaviour of the solutions near these points:

- the point $x_0 = 0$ is an **unstable** equilibrium point. If the solution starts near here, it will move away over time,
- the point $x_0 = a$ is an **stable** equilibrium point. If the solution starts near here, it will move towards this point over time.

1.4 Separable ODE

Definition 1.4.1

An ODE is **separable** if it can be written on the form

$$q(y(t))\frac{d}{dt}(y(t)) = p(t)$$

with known functions q and p.

The word separable comes from the ability to separate the parts depending on t and the parts depending on y on each side of the equality sign.

Example 1.4.1

The ODE

$$\dot{y} = -\sin(t)y + \sin(t)$$

is separable, since we can rewrite it as

$$\frac{\dot{y}}{1-y} = \sin(t)$$

If $y(t)$ is a solution of this, we get

$$\frac{y'(t)}{1-y(t)} = \sin(t)$$

Integrating with respect to t on both sides gives

$$\int \frac{y'(t)}{1-t}dt = \int \sin(t)dt$$

where the right hand side is recognized as $-\cos(t) + D_1$ for some constant C. What about the left hand side? Well, we can try to substitute $y(t)$ with y and get $y'(t)dt = dy$, so we have

$$\int \frac{1}{1-y} dy = -\ln|1-y| + D_2.$$

Hence, we have

$$\ln|1-y| = \cos(t) + D$$

where we have collected the constants into D, which gives us

$$1 - y = \pm e^D e^{\cos(t)}.$$

That is

$$y(t) = Ce^{\cos(t)} + 1$$

where we have put the pluss/minus sign and e^D into a new unknown C.

The above example also gave us the general solution strategy for solving separable ODE's.

Remark 1.4.1 Solution strategy

Given a separable ODE

$$q(y(t))\frac{d}{dt}(y(t)) = p(t)$$

1. Integrate with respect to t on both sides,

$$\int q(y(t))\frac{d}{dt}(y(t))\, dt = \int p(t)dt$$

2. Substitute $y = y(t)$ in the left integral, which gives $y'(t)dt = dy$,

$$\int q(y)dy = \int p(t)dt$$

3. Do the integration, and solve for y.

Problem Set - Differential equations

1. Verify that both

$$y_1(x) = 5e^x \text{ and } y_2(x) = 3e^x$$

solves

$$y'(x) = y(x)$$

Can you find more solutions? Sketch the solutions in the xy-plane.

2. Verify that both

$$y_1(t) = 3e^{-2t} \text{ and } y_2(t) = -2e^{3t}$$

are solutions of

$$y'' - y' - 6y = 0$$

Is $y(t) = 10y_1(t) + 2y_2(t)$ also a solution?

3. Determine for which real values $a \in \mathbb{R}$ the function $y(t) = \arctan(t)$ satisfy the differential equation

$$(t^2 + 1)y''(t) + aty'(t) = 0$$

4. Find the equilibrium points and classify them as either stable or unstable for the ODE

$$x'(t) = f(x)$$

where

a) $f(x) = x\left(\frac{1}{2} - e^{-|x|}\right)$

b) $f(x) = |x| - 1$

c) $f(x) = \sin(x)$

5. Solve the initial value problem

$$y' + \frac{2}{t}y = \frac{\cos(t)}{t^2}, \quad y\left(\frac{\pi}{2}\right) = 0$$

by using integrating factor.

6. Use separation of variables to solve the initial value problem

$$y' - t^2\sqrt{y} = 0, \quad y(0) = 1.$$

7. Solve the following ODEs

a) $y'(x) = \frac{3y-1}{x}$

b) $y'(x) + \frac{2y}{x} = \frac{1}{x^2}$

8. Find an equation for a curve going through the point $(2, 3)$ and which has slope $\frac{2x}{1+y^2}$.

Solutions - Differential equations

1. We find $y_1'(t)$ and $y_2'(t)$ and check if they satisfy the equation.

$$y_1'(x) = 5e^x$$

and we see that

$$y_1'(x) = 5e^x = y_1(x)$$

so y_1 is a solution of

$$y' = y$$

We also have

$$y_2'(x) = 3e^x = y_2(x)$$

so this is also a solution. In general, we have for every constant $C \in \mathbb{R}$ that $y = Ce^x$ is a solution of the equation.

2. We start by finding the first and second derivative of y_1 and y_2,

$$\begin{aligned} y_1'(t) &= -6e^{-2t} \\ y_1''(t) &= 12e^{-2t} \end{aligned}$$

and

$$\begin{aligned} y_2'(t) &= -6e^{3t} \\ y_2''(t) &= -18e^{3t} \end{aligned}$$

Now, we see that

$$\begin{aligned} y_1''(t) - y_1'(t) - 6y_1(t) &= 12e^{-2t} + 6e^{-2t} - 18e^{-2t} \\ &= e^{-2t}(12 + 6 - 18) \\ &= e^{-2t} \cdot 0 \\ &= 0 \end{aligned}$$

so y_1 solves the equation. Also

$$\begin{aligned} y_2''(t) - y_2'(t) - 6y_2(t) &= -18e^{3t} + 6e^{3t} + 12e^{3t} \\ &= e^{-2t}(-18 + 6 + 12) \\ &= e^{-2t} \cdot 0 \\ &= 0 \end{aligned}$$

so y_2 also solves the equation. Now, we can see that for any constants c_1 and c_2 we have that $y(t) = c_1y_1(t) + c_2y_2(t)$ gives us

$$\begin{aligned} y''(t) - y'(t) - 6y &= (c_1y_1(t) + c_2y_2(t))'' - (c_1y_1(t) + c_2y_2(t))' - 6(c_1y_1(t) + c_2y_2(t)) \\ &= c_1(y_1''(t) - y_1'(t) - 6y_1(t)) + c_2(y_2''(t) - y_2'(t) - 6y_2(t)) \\ &= 0 + 0 \\ &= 0 \end{aligned}$$

so in particular, if $c_1 = 10$ and $c_2 = 2$, we have that $y(t)$ is a solution of the ODE.

3. By differentiation, we get that

$$y'(t) = \frac{1}{1+t^2}, \quad y''(t) = -\frac{2t}{(1+t^2)^2},$$

so if y solves the differential equation, we get that

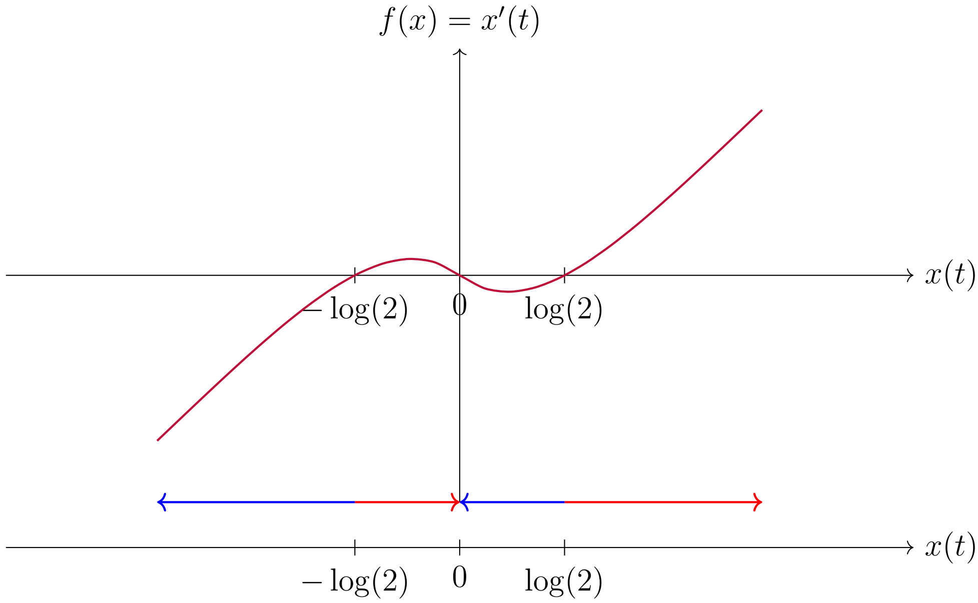
$$(t^2+1)\left(\frac{-2t}{(1+t^2)^2}\right) + at\frac{1}{1+t^2} = 0.$$

Thus,

$$\frac{-2t + at}{1 + t^2} = 0,$$

For this to be satisfied for all t, we see that $a = 2$.

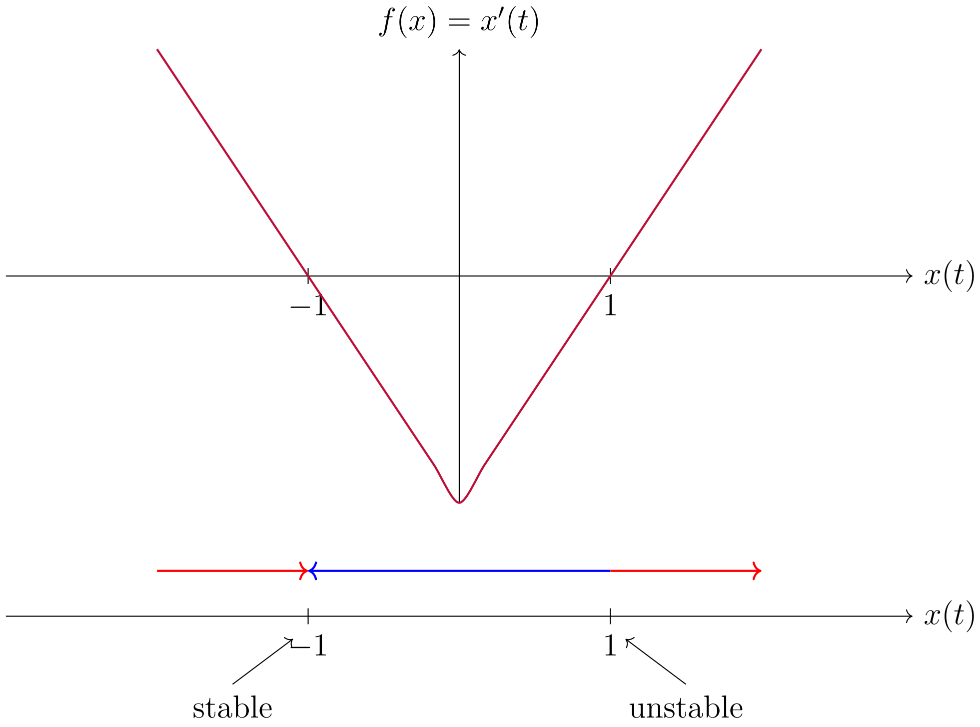
4.a) The function is zero in the points, $x = 0$ and $x = \pm\ln(2)$. These are the equilibrium points of the differential equation. Below is a drawing of f, and the qualitative behavior of the solutions to the differential equation.



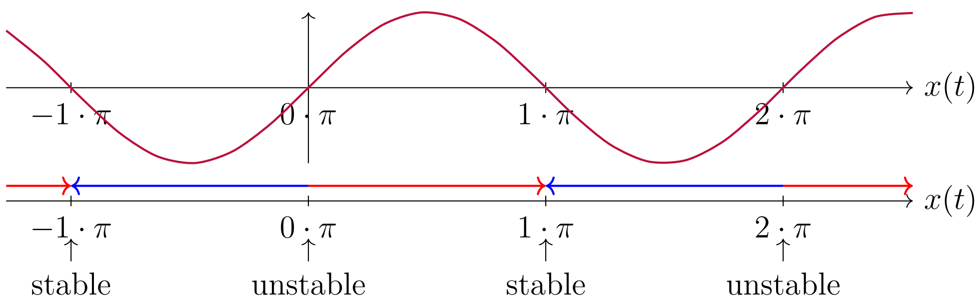
- If our initial data is on the interval $(-\infty, -\ln(2))$ the solution will go to ∞ as $t \to \infty$.
- On the interval $(-\ln(2), 0)$, the solution tends to the equilibrium point 0.
- On the interval $(0, \ln(2))$, the solution also tends to the equilibrium point 0.
- On the interval $(\ln(2), \infty)$, the solution tends to $+\infty$.

From the list above, we observe that the equilibrium point 0 is stable, while $\pm\ln(2)$ are unstable.

b)



c)



5. Let us for the sake of being lazy, assume that $t > 0$, then the integrating factor is

$$e^{F(t)} = e^{\int 2/t dt} = e^{2\ln(t)} = t^2$$

and we have

$$\begin{aligned} t^2y' + t^2\frac{2}{t}y &= t^2\frac{\cos(t)}{t^2} \\ \frac{d}{dt}\left(t^2y(t)\right) &= \cos(t) \\ t^2y(t) &= \int \cos(t)dt \\ t^2y(t) &= \sin(t) + C \\ y(t) &= \frac{\sin(t)+C}{t^2} \end{aligned}$$

since, we have the condition $y(\pi/2) = 0$, we need to have $C = -1$, so the solution of the initial value problem is

$$y(t) = \frac{\sin(t) - 1}{t^2}$$

Do note that this is only a solution for $t > 0$.

6. We write the equation as

$$y^{-1/2}y' = t^2,$$

and we proceed to integrate it

$$\int \frac{y'(t)}{\sqrt{y(t)}}dt = \int t^2 dt = \frac{1}{3}t^3 + C$$

On the left hand side we use the substitution $u = y(t)$, which gives us

$$du = y'(t)dt$$

and therefore the left hand side is

$$\int \frac{y'(t)}{\sqrt{y(t)}}dt = \int \frac{1}{\sqrt{u}}du = 2\sqrt{u} = 2y^{1/2}.$$

Thus, we have

$$2y^{1/2} = \frac{1}{3}t^3 + C$$

and therefore

$$y(t) = \left(\frac{C}{2} + \frac{1}{6}t^3\right)^2.$$

With the initial data we have

$$y(t) = \left(1 + \frac{1}{6}t^3\right)^2$$

7. a) We can rewrite the equation as

$$y' - \frac{3}{x}y = -\frac{1}{x}$$

The integrating factor is given by

$$e^{F(x)} = e^{\int -\frac{3}{x}dx} = e^{-3\ln|x|}.$$

This gives us

$$\frac{d}{dc}\left(e^{-3\ln|x|}y(x)\right) = -\frac{1}{x}e^{-3\ln|x|}.$$

Taking the integral on both sides, and using the substitution $u = -\ln|x|$, we obtain

$$e^{-3\ln|x|}y(x) = \int e^{-3\ln|x|}\left(-\frac{1}{x}\right)dx = \int e^{3u}du = \frac{1}{3}e^{3u} + C = \frac{e^{-3\ln|x|}}{3} + C$$

So, we have

$$y(x) = \frac{1}{3} + Ce^{3\ln|x|} = \frac{1}{3} + C|x|^3$$

Notice that the solutions before and after zero are independent since they are separated by zero. The solutions are therefore given by

$$y(x) = \begin{cases} 1/3 + C_1x^3, & x > 0 \\ 1/3 + C_2x^3, & x < 0 \end{cases}$$

b) The integrating factor is given by

$$e^{F(x)} = e^{\int 2/x dx} = 2\ln x = x^2$$

We multiply both sides of the equation with this and get

$$\frac{d}{dx}(x^2y) = 1.$$

We integrate both sides with respect to x and get

$$x^2y = x + C \implies y = \frac{1}{x} + \frac{C}{x^2}.$$

8. This is the same as solving the initial value problem

$$y' = \frac{2x}{1+y^2}, \quad y(2) = 3$$

which can be rewritten and integrated as

$$\int (1+y^2)dy = \int 2x dx.$$

This gives

$$y + \frac{y^3}{3} = x^2 + C_0$$

after rewriting

$$y^3 + 3y - 3x^2 = C$$

We use our initial data and have

$$C = 3^3 + 3 \cdot 3 - 3 \cdot 2^2 = 24,$$

so the sought after equation is therefore

$$y^3 + 3y - 3x^2 = 24.$$

Chapter 2

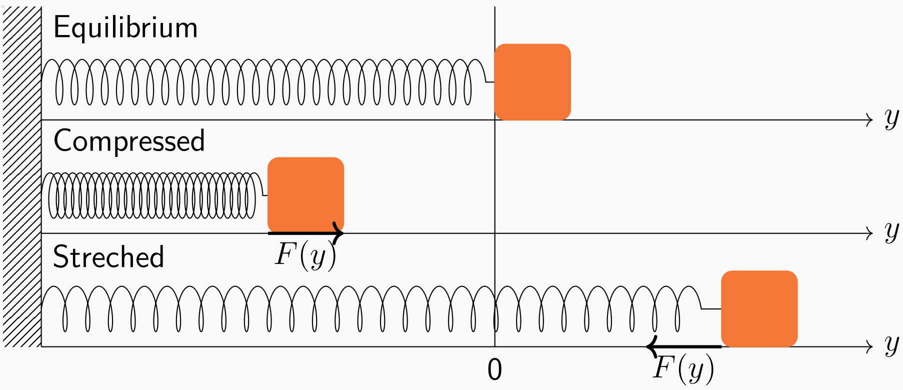
Second order equations

We now consider equations of the form

$$y''(t) + a_1(t)y'(t) + a_0(t)y(t) = f(t). \tag{2.1}$$

These are **second order linear ordinary differential equations**.

Example 2.0.1



A box connected to the wall by a spring is moving without friction along a surface. Hooke's law tells us that the restoring force in the spring is proportional to the displacement, that is

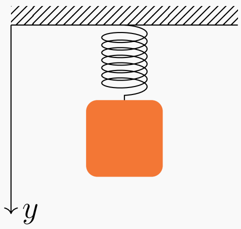
$$F(y) = -ky,$$

where y is the displacement of the box from the equilibrium state of the spring, k is the spring constant, and $F(y)$ is the force acting on the weight from the spring.

If $y(t)$ is the position of the box, then the acceleration of the box is $y''(t)$, and from Newton's second law of motion we have

$$-ky = my'',$$

where m is the mass of the box. This is a second order linear differential equation. Note that as any good physicist we have assumed that we work in a frictionless vacuum[a].



Now, let us assume that the box is connected to the ceiling. Then the gravitational pull will act with a constant force mg downwards. The total force is

$$F(y, y') = -ky - mg,$$

and Newton's second law of motion gives the ODE

$$my'' + ky = mg.$$

[a]https://xkcd.com/669/

In general we may not be able to find a solution to initial value problems of second order ODEs, but we will be working on **second order ODE's with constant coefficients**, which do give us unique solutions:

$$y''(t) + a_1 y'(t) + a_0 y(t) = f(t)$$

We will assume that y is two times differentiable, and defined on all real numbers. In practice, y will only be defined over some interval, but the methods will stay the same.

We will in addition assume that a_0 is nonzero. If one of them is zero, we can easily reduce the problem to a first order equation.

The ODE's are sorted into two categories the *homogeneous*

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

with zero on the right hand side, and the *inhomogeneous*

$$y''(t) + a_1 y'(t) + a_0 y(t) = f(t)$$

where $f(t)$ is a continuous function different from zero.

2.1 Homogeneous Equations

Differentiation is a linear operation, so we obtain at once a result which is often called the **superposition principle**.

Theorem 2.1.1 superposition principle

If $y_1(t)$ and $y_2(t)$ are both solutions to

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

then any scaled sum of them is also a solution, i.e. $c_1 y_1(t) + c_2 y_2(t)$ is a solution for all real numbers c_1 and c_2.

Proof. Let $y_1(t)$ and $y_2(t)$ be two solutions. Take any scaled sum of them $c_1 y_1(t) + c_2 y_2(t)$. Then

$$\begin{aligned} &(c_1 y_1(t) + c_2 y_2(t))'' \\ =&c_1 y_1''(t) + c_2 y_2''(t) \text{ differentiation is linear} \end{aligned}$$

Now use that both $y_1(t)$ and $y_2(t)$ are solutions:

$$\begin{aligned} &(c_1 y_1 + c_2 y_2)'' + a_1(c_1 y_1 + c_2 y_2)' + a_2(c_1 y_1 + c_2 y_2) \\ =&c_2(y_1'' + a_1 y_1' + a_2 y_1) + c_2(y_2'' + a_1 y_2' + a_2 y_2) \\ =&c_1 \cdot 0 + c_2 \cdot 0 = 0. \end{aligned}$$

The function $y(t) = c_1 y_1(t) + c_2 y_2(t)$ is also a solution. □

We have seen that as soon as we have one or two solutions to our equation, we can use the superposition principle to generate an infinite amount of them. Let us now find a way to generate these first two solutions. We try first with an exponential function $y(t) = e^{rx}$. Substituting this into the equation gives

$$\begin{aligned} 0 =&y''(t) + a_1 y'(t) + a_0 y(t) \\ =&r^2 e^{rt} + a_1 r e^{rt} + a_0 e^{rt} \\ =&e^{rt}(r^2 + a_1 r + a_0) \end{aligned}$$

Since e^{rt} is non-zero for all t and r, we see that we need $r^2 + a_1 r + a_0 = 0$. We call this quadratic equation the **characteristic polynomial** to the ODE, and we know that the roots are given by

$$r = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

We know that these roots may be on one of three forms: two real roots, one double root or two imaginary roots. Let us look at each of these cases separately.

Two real roots

From the discussion above we see that when we have two different real roots, r_1 and r_2, of the characteristic polynomial, we have the two solutions

$$y_1 = e^{r_1 t} \quad \text{and} \quad y_2 = e^{r_2 t}$$

From the superposition principle we also know that any scaled sum of these are also solutions to the ODE. In fact, we have that

$$y(t) = c_1 y_1(t) + c_2 y_2(t) = c_1 e^{r_1 t} + c_2 e^{r_2 t}$$

is the general solution of the ODE. We will not show that these are all the possible solutions though. Hopefully you trust me, or find some suitable references to consult.

Example 2.1.1

a) Find the general solution to

$$y'' - y' - 6y = 0$$

The characteristic polynomial of the equation is $r^2 - r - 6$ which have the roots

$$r = \frac{1 \pm \sqrt{1+24}}{2} = \begin{cases} -2 \\ 3 \end{cases}$$

so the general solution is

$$y = c_1 e^{-2t} + c_2 e^{3t}$$

b) Find the general solution to

$$y(t)'' - y(t) = 0.$$

The characteristic polynomial $r^2 - 1$ has the roots $r = 1$ and $r = -1$. The general solution is therefore

$$y(t) = c_1 e^{t} + c_2 e^{-t}.$$

Do notice that in the general solution we have two unknown values c_1 and c_2, so in order to find a solution to a initial value problem of a second order ODE we will need two conditions to get only one possible solution.

Example 2.1.2

a) Solve the initial value problem

$$y''(t) - y(t) = 0$$

with conditions

$$y(0) = 1, \quad y'(0) = 0.$$

We found the general solution

$$y(t) = c_1 e^t + c_2 e^{-t}.$$

and after differentiation

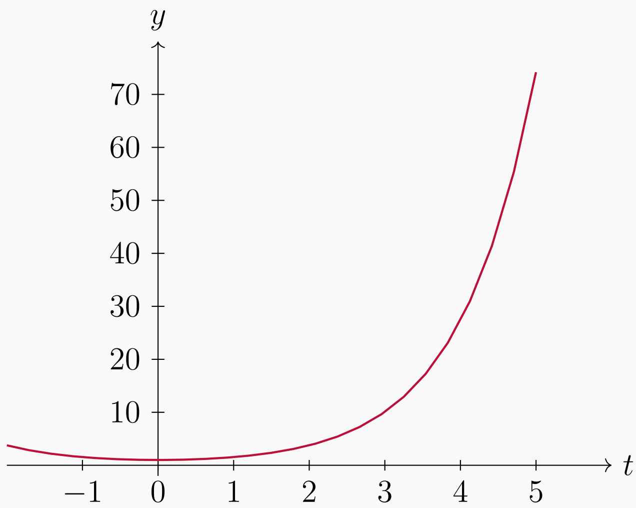
$$y'(t) = c_1 e^t - c_2 e^{-t}.$$

When imposing the conditions on this, we obtain the following two equations

$$\begin{array}{ccccccc} 1 & = & y(0) & = & c_1 & + & c_2 \\ 1 & = & y'(0) & = & c_1 & - & c_2 \end{array}$$

Solving these gives $c_1 = \frac{1}{2}$ and $c_2 = \frac{1}{2}$. The solution is

$$y(t) = \frac{1}{2}e^t + \frac{1}{2}e^{-t}.$$



b) Solve the initial value problem

$$y'' - y' - 6y = 0$$

with conditions

$$y(0) = 2, \quad y'(0) = 1.$$

We found the general solution

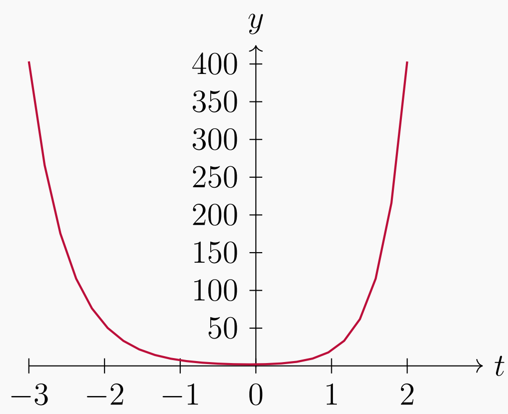
$$y = c_1 e^{-2t} + c_2 e^{3t}$$

and obtain the equations

$$\begin{array}{ccccccc} 2 & = & y(0) & = & c_1 & + & c_2 \\ 1 & = & y'(0) & = & -2c_1 & + & 3c_2 \end{array}$$

Which gives $c_1 = 1$ and $c_2 = 1$, so the solutions is

$$y(t) = e^{-2t} + e^{3t}$$



Double root

If we have only one real root r_1, then we only get one solution from the characteristic polynomial, namely

$$y_1 = e^{r_1 t}.$$

To get a general solution to the problem we need to find another, independent solution as well. We won't dwell to long on what this other solution may be, but instead reveal that it is $y_2 = te^{r_1 t}$. To verify this we observe that since r_1 is the only root of the polynomial, we have

$$r^2 + a_1 r + a_0 r = (r - r_1)^2 = r^2 - 2r_1 r + r_1^2$$

so $a_1 = -2r_1$ and $a_0 = r_1^2$. The ODE can thus be written as

$$y'' - 2r_1 y' + r_1^2 y = 0$$

Before substituting y_2 on the left we calculate the derivatives of y_2,

$$\begin{aligned} y_2' &= e^{r_1 t} + r_1 t e^{r_1 t} \\ y_2'' &= 2r_1 e^{r_1 t} + r_1^2 t e^{r_1 t}. \end{aligned}$$

Now, we get

$$\begin{aligned} y_2'' - 2r_1 y_2' + r_1 y_2 &= (te^{r_1 t})'' - 2r_1 (te^{r_1 t})' + r_1^2 e^{r_1 t} \\ &= 2r_1 e^{r_1 t} + r_1^2 t e^{r_1 t} - 2r_1(e^{r_1 t} + r_1 t e^{r_1 t}) + r_1^2 e^{r_1 t} \\ &= 0 \end{aligned}$$

We now have two solutions

$$y_1 = e^{r_1 t} \quad \text{and} \quad y_2 = te^{r_1 t}$$

to the ODE, and as with the case of two real roots we tell at once that

$$y(t) = c_1 e^{r_1 t} + c_2 t e^{r_1 t}$$

is the general solution. Those who can't accept this without a proof is once again asked to consult some suitable reference.

Example 2.1.3

a) Find the general solution to

$$y(t)'' + 2y(t)' + y(t) = 0.$$

The characteristic equation is

$$r^2 + 2r + 1$$

hhich only has one root $r = -1$. Then the general solution is

$$y(t) = c_1 t e^{-t} + c_0 e^{-t}.$$

b) Solve the initial value problem

$$y'' - 4y' + 4y = 0$$

with the conditions

$$y(1) = 0, \quad y'(1) = 2$$

$$r = \frac{4 \pm \sqrt{16 - 16}}{2} = 2$$

The general solution is

$$y(t) = c_1 e^{2t} + c_2 t e^{2t}$$

and the derivative of this is

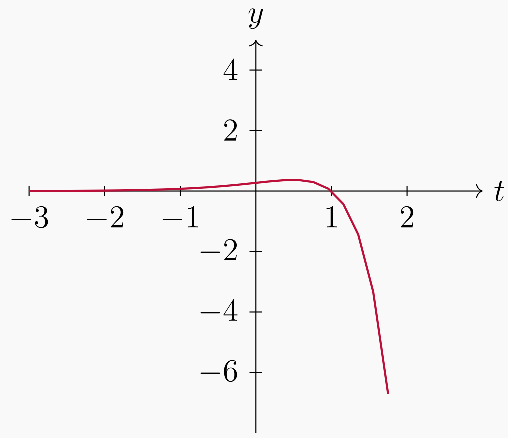
$$y'(t) = 2c_1 e^{2t} + c_2 e^{2t} + 2c_2 t e^{2t} = 2c_1 e^{2t} + c_2(1 + 2t)e^{2t}$$

We obtain the following equations

$$\begin{array}{ccccccc} 0 & = & y(1) & = & c_1 e^2 & + & c_2 e^2 \\ 2 & = & y'(1) & = & 2c_1 e^2 & + & 3c_2 e^2 \end{array}$$

which solves to $c_1 = 2e^{-2}$ and $c_2 = -2e^{-2}$. The solution of the initial value problem is therefore

$$y(t) = 2e^{2t-2} - 2te^{2t-2}$$



c) Solve the initial value problem

$$y'' + y' + 0.25y = 0$$

with the conditions

$$y(0) = 3, \quad y'(0) = -3.5$$

$$r = \frac{-1 \pm \sqrt{1 - 4 \cdot 0.25}}{2} = -\frac{1}{2} = -0.5$$

The general solution is

$$y(t) = c_1 e^{-0.5t} + c_2 t e^{-0.5t}$$

and the derivative of this is

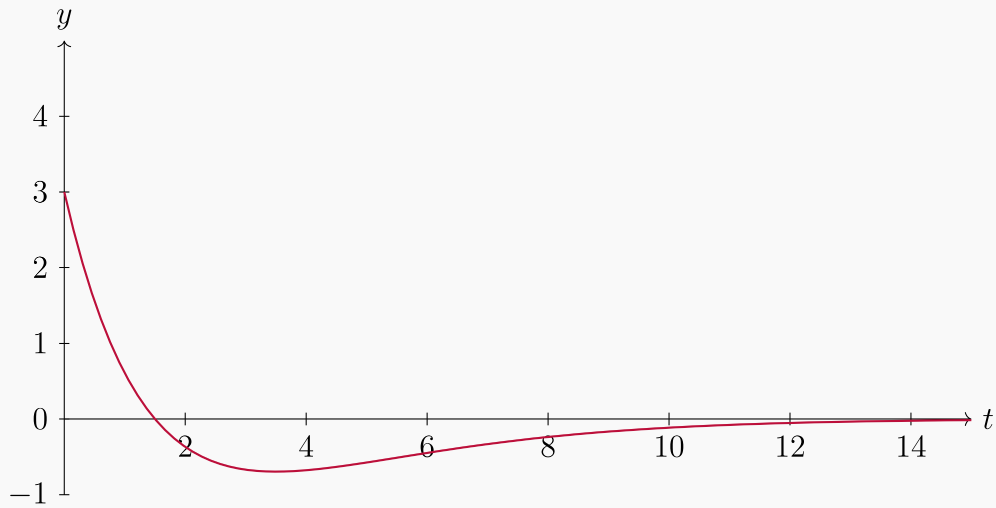
$$y'(t) = c_2 e^{-0.5t} - 0.5(c_1 + c_2 t)e^{-0.5t}.$$

Substituting $t = 0$ and using the initial condition we get

$$y(0) = c_1 = 3, \quad y'(0) = c_2 - 0.5c_1 = -3.5 \quad \Longrightarrow \quad c_2 = -2$$

The solution of the initial value problem is therefore

$$y(t) = 3e^{-0.5t} - 2te^{-0.5t}$$



Two complex roots

Now, the last case is two complex roots. We have implicitly assumed throughout that our differential equation is real-valued and specifically that a_1 and a_0 are real numbers. Then we know that the roots are complex conjugate of each other,

$$r_1 = a + ib, \quad r_2 = a - ib$$

We know that

$$e^{(a+ib)t} = e^{at}e^{ibt}, \quad \text{and} \quad e^{(a-ib)t} = e^{at}e^{-ibt}$$

are solutions to our ODE, but as we are mostly interested in real-valued solutions we want to dismiss the complex ones. The first step to do this is to recall **Euler's formula** which tells us that

$$e^{ib} = \cos(b) + i\sin(b).$$

After a bit of algebraic manipulation we see that

$$\frac{e^{ibt} + e^{-ibt}}{2} = \cos(bt), \quad \text{and} \quad \frac{e^{ibt} - e^{-ibt}}{2i} = \sin(bt).$$

From this, we obtain the following two real-valued solutions

$$\begin{aligned} y_1(t) &= \frac{1}{2}e^{(a+ib)t} + \frac{1}{2}e^{(a-ib)t} \\ &= e^{at}\frac{e^{ib} + e^{-ibt}}{2} \\ &= e^{at}\cos(bt) \end{aligned}$$

and

$$\begin{aligned} y_2(t) &= \frac{1}{2i}e^{(a+ib)t} - \frac{1}{2i}e^{(a-ib)t} \\ &= e^{at}\frac{e^{ib} - e^{-ibt}}{2i} \\ &= e^{at}\sin(bt) \end{aligned}$$

The general real-valued solution of the ODE having complex roots $a \pm ib$ to the characteristic polynomial is on the form

$$y(t) = e^{at}[c_1\cos(bt) + c_2\sin(bt)].$$

Once again, if you want a proof that these are all the real-valued solution you should consult some suitable reference.

Example 2.1.4

a) Find the general real-valued solution of

$$y(t)'' + y(t) = 0.$$

The characteristic polynomial $r^2 + 1$ have the complex roots $r = i$ and $\overline{r} = -i$.

That is $a = 0$ and $b = 1$, so the general solution is

$$y(t) = c_1 \cos t + c_2 \sin t.$$

b) Find the general solution of

$$y'' + 2y' + 4y = 0$$

The characteristic polynomial $r^2 + 2r + 4$ have the complex roots $r = -1 \pm i\sqrt{3}$. That is $a = -1$ and $b = \sqrt{3}$, so the general solution is

$$y(t) = e^{-t}[c_1 \cos(\sqrt{3}t) + c_2 \sin(\sqrt{3}t)]$$

c) Solve the initial value problem

$$y'' + 0.4y' + 9.04y = 0$$

with the conditions

$$y(0) = 0, \quad y'(0) = -3$$

$$r = \frac{-0.4 \pm \sqrt{0.16 - 36.16}}{2} = \begin{cases} -0.2 + 3i \\ -0.2 - 3i \end{cases}$$

That is, $a = -0.2$ and $b = 3$, so the general solution is

$$y(t) = e^{-0.2t}[c_1 \cos(3t) + c_2 \sin(3t)]$$

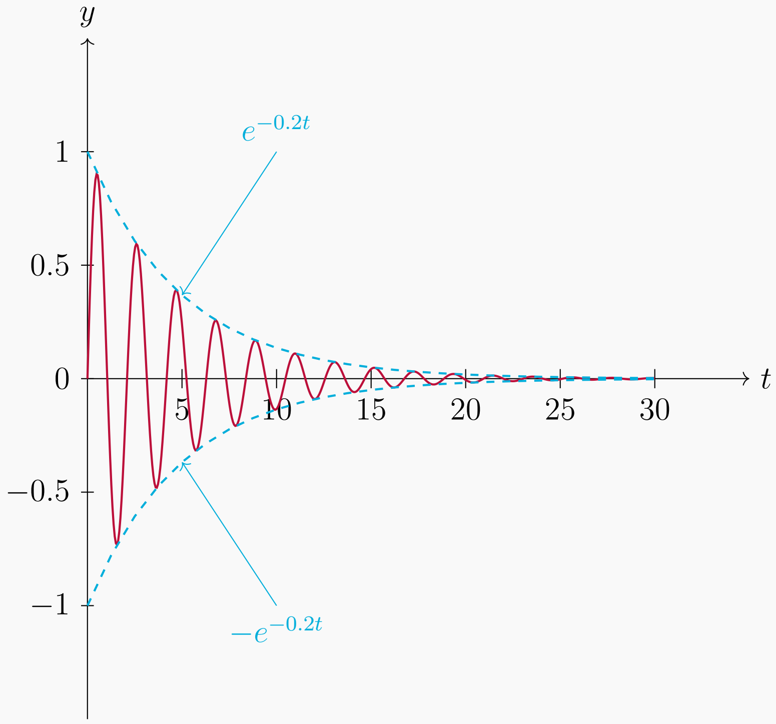
At once we observe that the condition $y(0) = 0$ forces $c_1 = 0$ since $\cos(0) = 1$. To determine c_2 we know differentiate the resulting expression

$$y'(t) = c_2[-0.2e^{-0.2t} \sin(3t) + 2e^{-0.2t} \cos(3t)].$$

By substituting in $t = 0$ and using the second condition we get $y'(0) = 3c_2 = 3$ sp $c_2 = 1$. The particular solution is

$$e^{-0.2t} \sin(3t).$$

$$y(t) = 3e^{-0.5t} - 2te^{-0.5t}$$



c) Solve the initial value problem

$$y'' - 0.2y' + 16.01y = 0$$

with the conditions

$$y(0) = 1, \quad y'(0) = 4.1$$

$$r = \frac{0.2 \pm \sqrt{0.04 - 4 \cdot 16.01}}{2} = \begin{cases} 0.1 + 4i \\ 0.1 - 4i \end{cases}$$

That is, $a = 0.1$ and $b = 4$, so the general solution is

$$y(t) = e^{0.1t}[c_1 \cos(4t) + c_2 \sin(4t)]$$

From the condition $y(0) = 1$ we see that $c_1 = 0$, so we have

$$y(t) = e^{0.1t}[\cos(4t) + c_2 \sin(4t)]$$

and in order to obtain c_2 we differentiate:

$$y'(t) = e^{0.1t}[(4c_2 + 0.1)\cos(4t) + (0.1c_2 - 4)\sin(t)].$$

Substituting $t = 0$ we get

$$y'(0) = 4c_2 + 0.1 = 4.1$$

so $c_2 = 1$ and the particular solution is

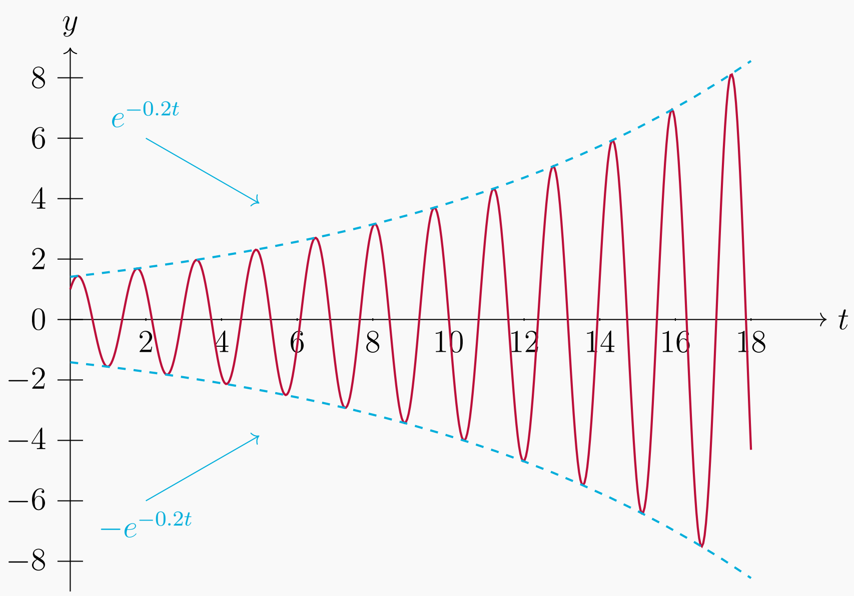
$$y = e^{0.1t}[\cos(4t) + \sin(4t)]$$

Now, since the cosine and sine has the same period given by 4, we can rewrite the expression in the bracket as a cosine with some amplitude and a phase shift, i.e.

$$\cos(4t) + \sin(4t) = A\cos(4t - \phi)$$

for some amplitude A and some phase ϕ. As will be remarked in the summary the amplitude A is given by $\sqrt{1^2 + 1^2} = \sqrt{2}$, and the phase ϕ is given by $\arctan(1/1) = \pi/4$, hence our solution can also be written as

$$y(t) = \sqrt{2}e^{0.1t}\cos(4t - \pi/4)$$



Summary

Theorem 2.1.2

The general solution of

$$y'' + a_1y' + a_0 = 0$$

depends on the roots of the characteristic polynomial

$$r^2 + a_1r + a_0.$$

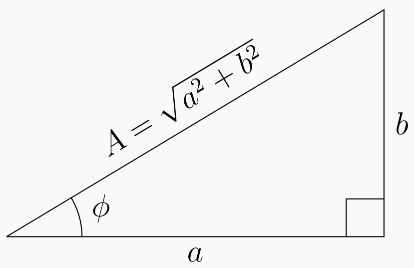
The general solution is

$$y(t) = \begin{cases} c_1e^{r_1t} + c_2e^{r_2t} & \text{if there are two real roots } r_1 \text{ and } r_2. \\ c_1e^{r_1t} + c_2te^{r_1t} & \text{if there are is a double root } r_1. \\ e^{at}[c_1\cos(bt) + c_2\sin(bt)] & \text{if there are two imaginary roots } a \pm ib. \end{cases}$$

Remark 2.1.1

If we get a particular solution to a second order ODE consisting of some exponential and a function on the form $a\cos(ct) + b\sin(ct)$ we will often like to rewrite this part to get a single cosine-function $A\cos(ct - \phi)$.

We do this by first remembering that if we have a right triangle with catheti (or more commonly; legs), of length a and b



then the hypotenuse has length $\sqrt{a^2+b^2} = A$ by the Pythagorean Theorem, and the angle between the hypotenuse and the leg of length a has angle ϕ given by $\phi = \arctan\frac{b}{a}$. From the definition of cosine and sine, we have

$$a = A\cos(\phi) \quad \text{and} \quad b = A\sin(\phi).$$

Now, using the identity $\cos(u \pm v) = \cos(u)\cos(v) \mp \sin(u)\sin(v)$ we observe that

$$a\cos(ct) + b\sin(ct) = A\cos(\phi)\cos(ct) + A\sin(\phi)\sin(ct) = A\cos(ct - \phi)$$

However neither a nor b is necessarily positive, but the formulas will hold true up to possibly a difference of π in the angle ϕ, i.e. the angle ϕ lie in the same quadrant as the point (a, b). To summarize

$$a\cos(ct) + b\sin(ct) = A\cos(ct - \phi)$$

where $A = \sqrt{a^2 + b^2}$ and $\phi = \arctan\frac{b}{a}\ (+\pi)$ lie in the same quadrant as (a, b).

2.2 Inhomogeneous equations

Let us now look at the inhomogeneous case, that is equations on the form

$$y''(t) + a_1 y'(t) + a_0 y(t) = f(t) \tag{2.2}$$

where $f(t)$ is a continuous function, not everywhere equal to zero. We will write $y_p(t)$ for solutions to (2.2) and call them partical solutions.

Assume that we have found a particular solution $y_p(t)$. If $y(t)$ is some other solution to (2.2), then $y(t) - y_p(t)$ is a homogeneous solution, that is, a solution to the corresponding homogeneous equation,

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0.$$

Theorem 2.2.1

Every solution to the inhomogeneous equation is on the form

$$y(t) = y_p(t) + y_h(t)$$

where $y_p(t)$ is a particular solution and $y_h(t)$ is a solution to the corresponding homogeneous equation.

In order to find all the solutions we therefore get the following strategy: find one particular solution and then add the general solution of the homogeneous case.

Finding a particular solution

There is an analytic way to find a particular solution which we will give shortly. However, there is a much easier method that works for a surprising amount of cases which we will focus on.

Remark 2.2.1 Analytic method / Method of variation of parameters

Let $y_1(t)$ and $y_2(t)$ be two independent solutions to the homogeneous equation. Then the particular solution can be found as

$$\begin{aligned} y_p(t) = y_2(t) \int \frac{y_1(t) f(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}\, dt \\ - y_1(t) \int \frac{y_2(t) f(t)}{y_1(t)y_2'(t) - y_2(t)y_1'(t)}\, dt. \end{aligned}$$

Now, let us move on to the method we love and adore: **The method of undetermined coefficients**.

The idea: Look for a solution that looks like the right hand side function $f(t)$ of (2.2) with general coefficients. Determine these coefficients by substituting this into the problem. If the right hand side is of the same form as the solution to the homogeneous problem, multiply by t as in the case with a single root.

This method relies on you seeing examples and know what to look for in each case. It really is a method relying on intuition.

Example 2.2.1

Let $f(t) = K \neq 0$ be a constant and consider the inhomogeneous ODE

$$y''(t) + a_1 y'(t) + a_0 y(t) = K$$

with $a_0 \neq 0$ not zero. We try with a particular solution $y_p(t) = c$ which is also a constant. Substituting into the equation

$$K = y_p''(t) + a_1 y_p'(t) + a_0 y_p(t) = a_0 c.$$

Since $a_0 \neq 0$, we get the particular solution $y_p(t) = K/a_0$.

Example 2.2.2

We consider the equation

$$y'' + 2y' + 2y = -2e^{-t} \sin t. \tag{2.3}$$

The characteristic polynomial is

$$r^2 + 2r + 2$$

and so the roots are given by completing the square,

$$(r+1)^2 + 1 = 0.$$

So we have two roots $r_\pm = -1 \pm i$. The basis for the homogeneous problem solution space is thus

$$y_1(t) = e^{-t}\cos t, \text{ and } y_2(t) = e^{-t}\sin t.$$

If the right hand side was not already part of the solution for the homogeneous problem, we would try for the particular solution

$$y_p(t) = Ae^{-t}\cos t + Be^{-t}\sin t$$

and determine A and B. Instead we will try

$$y_p(t) = Ate^{-t}\cos t + Bte^{-t}\sin t.$$

Substituting this into (2.4), we find that

$$2e^{-t}(B\cos(t) - A\sin(t)) = -2e^{-t}\sin t,$$

and so we find $B = 0$ and $A = 1$. Thus the general solution is

$$\begin{aligned} y(t) &= y_p(t) + c_1 y_1(t) + c_2 y_2(t) \\ &= te^{-t}\cos t + c_1 e^{-t}\cos t + c_2 e^{-t}\sin t, \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

Example 2.2.3

Consider the ODE

$$y''(t) + y(t) = e^t.$$

We try $y_p(t) = ce^t$ with a constant c. Substituting $y_p(t)$ in the equation, tells us that $c = 1/2$.

Example 2.2.4

We consider the inhomogeneous ODE

$$y''(t) - y(t) = \cos t.$$

We try $y_p(t) = a\cos t + b\sin t$ with constants a and b. Note that we have to include both $\cos t$ and $\sin t$ even when $f(t)$ only contains $\cos t$. Substituting $y_p(t)$ in the equation:

$$\begin{aligned}
\cos t &= y_p''(t) - y_p(t) \\
&= (a\cos t + b\sin t)'' - (a\cos t + b\sin t) \\
&= -a\cos t - b\sin t - a\cos t - b\sin t \\
&= -2a\cos t - 2b\sin t.
\end{aligned}$$

The left and right hand side is only equal for all t if $a = -\frac{1}{2}$ and $b = 0$. The general solution to the inhomogeneous equation is

$$y(t) = c_1 te^t + c_2 e^{-t} - \frac{1}{2}\cos t.$$

Example 2.2.5

Consider the inhomogeneous ODE

$$y''(t) + 2y'(t) + y(t) = t^2 - 1.$$

The right hand side $f(t)$ is a polynomial. Thus we try with a polynomial of the same degree as $f(t)$: $y_p(t) = at^2 + bt + c$ with constants a, b, c. Note that we include the part bt even though it do not appear in $f(t)$. Substitute $y_p(t)$ in the equation:

$$\begin{aligned}
t^2 - 1 &= y_p''(t) + 2y_p(t) + y_p(t) \\
&= 2a + 2(2at + b) + (at^2 + bt + c) \\
&= 2a + 4at + 2b + at^2 + bt + c \\
&= at^2 + (4a + b)t + 2a + 2b + c.
\end{aligned}$$

The left and right hand side are only equal if

$$\begin{aligned}
1 &= a \\
0 &= 4a + b \\
-1 &= 2a + 2b + c.
\end{aligned}$$

Hence $a = 1$, $b = -4$ and $c = 5$, that is

$$y_p(t) = t^2 - 4t + 5.$$

The general solution of the inhomogeneous ODE is therefore on the form

$$y(t) = c_1 te^{-t} + c_2 e^{-t} + t^2 - 4t + 5.$$

Example 2.2.6

We consider the equation

$$y'' + 2y' + 2y = -2e^{-t} \sin t. \tag{2.4}$$

The characteristic polynomial is

$$r^2 + 2r + 2$$

and so the roots are given by completing the square,

$$(r+1)^2 + 1 = 0.$$

So we have two roots $r_\pm = -1 \pm i$. The basis for the homogeneous problem solution space is thus

$$y_1(t) = e^{-t} \cos t, \text{ and } y_2(t) = e^{-t} \sin t.$$

If the right hand side was not already part of the solution for the homogeneous problem, we would try for the particular solution

$$y_p(t) = Ae^{-t} \cos t + Be^{-t} \sin t$$

and determine A and B. Instead we will try

$$y_p(t) = Ate^{-t} \cos t + Bte^{-t} \sin t.$$

Substituting this into (2.4), we find that

$$2e^{-t}(B\cos(t) - A\sin(t)) = -2e^{-t} \sin t,$$

and so we find $B = 0$ and $A = 1$. Thus the general solution is

$$\begin{aligned} y(t) &= y_p(t) + c_1 y_1(t) + c_2 y_2(t) \\ &= te^{-t} \cos t + c_1 e^{-t} \cos t + c_2 e^{-t} \sin t, \end{aligned}$$

where c_1 and c_2 are arbitrary constants.

Example 2.2.7

Consider the equation

$$y''(t) + 2y'(t) + y(t) = e^{-t}.$$

We can't have the particular solution $y_p(t) = ce^{-t}$, since it is also a solution to the homogeneous equation:

$$\begin{aligned} y''(t) + 2y'(t) + y(t) &= (ce^{-t})'' + 2(ce^{-t})' + ce^{-t} \\ &= ce^{-t} - 2ce^{-t} + ce^{-t} \\ &= 0. \end{aligned}$$

We tried in the example above to add on a factor t to remedy this, but here this is also a solution of the homogeneous equation. Then we add on another factor t and try $y_p(t) = ct^2e^{-t}$:

$$\begin{aligned} e^{-t} &= y_p''(t) + 2y_p'(t) + y_p(t) \\ &= (ct^2e^{-t})'' + 2(ct^2e^{-t})' + ct^2e^{-t} \\ &= ce^{-t}(t^2 - 4t + 2 + 4t - 2t^2 + t^2) \\ &= 2ce^{-t}. \end{aligned}$$

since $e^{-t} \neq 0$ for all t, we need c to be equal to $1/2$. We are left with $y_p(t) = \frac{1}{2}t^2e^{-t}$. The general solution of the inhomogeneous ODE is then on the form

$$y(t) = c_1e^{-t} + c_2te^{-t} + \frac{1}{2}t^2e^{-t}.$$

Even though this method is to some degree based on intuition, we list a few good guiding rules here.

Remark 2.2.2

1. **Basic rule:** If $f(t)$ in (2.2) is one of the functions in the first column in Table 2.2, choose the trial solution $y_p(t)$ in the second column.

2. **Modification:** If a term of you choice for $y_p(t)$ is a solution of the homogeneous ODE, multiply it with t. If the solution correspond to a double root, multiply with t^2.

3. **Sum rule:** If $f(t)$ is a sum of functions from the first column of Table 2.2, choose $y_p(t)$ as a sum of the corresponding trial solutions.

Forcing function $f(t)$	Trial solution $y_p(t)$	Comment
Ke^{ct}	ae^{ct}	
$K\cos(\omega t)$ or $K\sin(\omega t)$	$a\cos(\omega t) + b\sin(\omega t)$	
$p(t)$	$q(t)$	p is a polynomial; q is a polynomial of same degree
$p(t)\cos(\omega t)$ or $p(t)\sin(\omega t)$	$q(t)\cos(\omega t) + r(t)\sin(\omega t)$	p is a polynomial; q & r are polynomials of same degree
$Ke^{ct}\cos(\omega t)$ or $Ke^{ct}\sin(\omega t)$	$e^{ct}[a\cos(\omega t) + b\sin(\omega t)]$	
$e^{ct}p(t)\cos(\omega t)$ or $e^{ct}p(t)\sin(\omega t)$	$e^{ct}q(t)\cos(\omega t) + e^{ct}r(t)\cos(\omega t)$	p is a polynomial; q & r are polynomials of same degree

Table 2.1: Method of Undetermined Coefficients

Problem Set - Second order ODEs

1. Find the general solution of the equations

a) $y'' - y' - 2y = 0$

b) $y'' + y = 0$

2. What is the general solution of the equation

$$y'' - 3y' + 2y = e^{3t}$$

3. The equation

$$y'' + 2y' + 2y = -2e^{-t}\sin(t),$$

has a particular solution on the form

$$y_p(t) = Ate^{-t}\cos(t) + Bte^{-t}\sin(t)$$

Show that this is in fact a solution if $A = 1$ and $B = 0$
(plug the expression $y_p(t) = te^{-t}\cos(t)$ into the equation).

4. Find a particular solution to each of the equations

a) $y'' - y' - 2y = te^t$

b) $y'' + y = \cos(t)$

5. Solve the initial value problems

a) $y''-y'-2y = 0, \quad y(0) = 0$ and $y'(0) = 1$

b) $y'' + y = 0, \quad y\left(\frac{\pi}{2}\right) = 1$ and $y'\left(\frac{\pi}{2}\right) = 0$

6) Solve the initial value problem

$y'' - 4y' + 13y = 0, \quad y(0) = 1$ and $y'(0) = 5.$

Rewrite the solution to the form

$$y(t) = Ae^{at}\cos(bt - \phi)$$

Solutions - Second order ODEs

1.a The characteristic polynomial is

$$r^2 - r - 2$$

which has the roots

$$r = \frac{1 \pm \sqrt{1 - 4 \cdot (-2)}}{2} = \frac{1 \pm 3}{2} = \begin{cases} 2 \\ -1 \end{cases}$$

so the general solution is

$$y_h(t) = c_1 e^{2t} + c_2 e^{-t}$$

1.b The characteristic polynomial is

$$r^2 + r = 0$$

which has the roots

$$r = \frac{0 \pm \sqrt{0 - 4}}{2} = \pm i$$

so the general solution is given by

$$y_h(t) = e^{0t}[c_1 \cos(t) + c_2 \sin(t)] = c_1 \cos(t) + c_2 \sin(t)$$

2. The characteristic polynomial is

$$r^2 - 3r + 2$$

which has roots

$$r = \frac{3 \pm \sqrt{9 - 8}}{2} = \frac{3 \pm 1}{2} = \begin{cases} 2 \\ 1 \end{cases}$$

so the general homogeneous solution is

$$y_h(t) = c_1 e^{2t} + c_2 e^{t}$$

Now, since we observe that e^{3t} do not solve the homogeneous equation, we try to solve the non-homogeneous equation with a particular solution on the form

$$y_p(t) = Ae^{3t}$$

We put this into the equation and get

$$\begin{aligned} y_p''(t) - 3y_p'(t) + 2y_p(t) &= 9Ae^{3t} - 9Ae^{3t} + 2Ae^{3t} \\ &= 2Ae^{3t} \end{aligned}$$

which tells us that y_p is a particular solution if $A = \frac{1}{2}$. Thus, the general solution is

$$y(t) = y_h(t) + y_p(t) = c_1e^{2t} + c_2e^t + \frac{1}{2}e^{3t}$$

3. We start by finding the derivatives

$$\begin{aligned} y_p(t) =& te^{-t}\cos(t) \\ y_p'(t) =& e^{-t}\cos(t) - te^{-t}\cos(t) - te^{-t}\sin(t) \\ y_p''(t) =& -e^{-t}\cos(t) - e^{-t}\sin(t) - e^{-t}\cos(t) + te^{-t}\cos(t) + te^{-t}\sin(t) \\ & - e^{-t}\sin(t) + te^{-t}\sin(t) - te^{-t}\cos(t) \\ =& -2e^{-t}\cos(t) - 2e^{-t}\sin(t) + 2te^{-t}\sin(t), \end{aligned}$$

and therefore

$$\begin{aligned} y_p''(t) + 2y_p'(t) + 2y_p(t) =& -2e^{-t} - 2e^{-t}\sin(t) + 2te^{-t}\sin(t) \\ & + 2e^{-t}\cos(t) - 2te^{=t}\cos(t) - 2te^{-t}\sin(t) \\ & + 2te^{-t}\cos(t) \\ =& -2e^{-t}\sin(t) \end{aligned}$$

which shows that y_p is a solution of the equation.

4.a We know from 1.a) that the general solution of the homogeneous case is given by

$$y_h(t) = c_1e^{2t} + c_2e^{-t}$$

and since te^t can't be found as a solution of this, we will try with a first degree polynomial times e^t as our particular solution,

$$\begin{aligned} y_p(t) =& (At + B)e^t \\ y_p'(t) =& Ate^t + Ae^t + Be^t \\ =& Ate^t + (A + B)e^t \\ y_p''(t) =& Ate^t + Ae^t + (A + B)e^t \\ =& Ate^t + (2A + B)e^t \end{aligned}$$

We put this into the equation and get

$$\begin{aligned} y''(t) - y'(t) - 2y(t) &= (Ate^t + (2A + B)e^t) - (Ate^t + (A + B)e^t) - 2(Ate^t + Be^t) \\ &= -2Ate^t + (2A + B - A - B - 2B)e^t. \end{aligned}$$

We know that this should be equal $te^t + 0$, so we obtain the equations

$$\begin{aligned} -2A =& 1 \\ 0 =& 2A + B - A - B - 2B \\ =& A - 2B \end{aligned}$$

by comparing the two expressions. The first equation tells us that $A = -\frac{1}{2}$, and after putting this into the second equation we get $B = \frac{A}{2} = -\frac{1}{4}$. The particular solution is

$$y_p(t) = -\frac{1}{2}te^t - \frac{1}{4}e^t$$

We could also evaluate the equation

$$te^t = -2Ate^t + (2A + B - A - B - 2B)e^t$$

for two values of t, say $t = 0$ and $t = 1$ to obtain two equations

$$\begin{aligned} t = 0: &\quad 0 = 2A + B - A - B - 2B \\ t = 1: &\quad e = -2Ae + (2A + B - A - B - 2B)e \end{aligned}$$

and solve these two for A and B.

4.b We know from 1.b) that the homogeneous solution is given by

$$y_h(t) = c_1 \cos(t) + c_2 \sin(t)$$

and since $\cos(t)$ is a homogeneous solution, we try the particular solution

$$\begin{aligned} y_p(t) &= At\cos(t) + Bt\sin(t) \\ y_p'(t) &= A\cos(t) - At\sin(t) + B\sin(t) + Bt\cos(t) \\ &= (A + Bt)\cos(t) + (B - At)\sin(t) \\ y_p''(t) &= B\cos(t) - (A + Bt)\sin(t) - A\sin(t) + (B - At)\cos(t) \\ &= (2B - At)\cos(t) - (2A + Bt)\sin(t) \end{aligned}$$

We put this into the equation and get

$$\begin{aligned} y'' + y =& (2B - At)\cos(t) - (2A + Bt)\sin(t) + At\cos(t) + Bt\sin(t) \\ =& 2B\cos(t) - 2A\sin(t) = \cos(t) \end{aligned}$$

which tells us that $A = 0$ and $B = \frac{1}{2}$. Our particular solution is therefore

$$y_p(t) = \frac{1}{2}t\sin(t)$$

5.a We know that the general solution is

$$y_h(t) = c_1 e^{2t} + c_2 e^{-t}$$

We use the information to find the equations

$$\begin{aligned} c_1 + c_2 &= y_h(0) = 0 \\ 2c_1 - c_2 &= y_h'(0) = 1 \end{aligned}$$

By combining these two equations we get that $2c_1 + c_1 = 1$, that is $c_1 = \frac{1}{3}$. This in turn gives us $c_2 = -\frac{1}{3}$. The solution of the IVP is therefore

$$y(t) = \frac{1}{3}(e^{2t} - e^{-t})$$

5.b We know that the general solution is

$$y_h(t) = c_1 \cos(t) + c_2 \sin(t)$$

This tells us that $y(\pi/2) = c_2 = 1$. After differentiating we have

$$y'\left(\frac{\pi}{2}\right) = -c_1 \sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) = -c_1 = 0$$

Thus our solution to the IVP is

$$y(t) = \sin(t)$$

6. The characteristic polynomial is

$$r^2 - 4r + 13$$

which has roots

$$r = \frac{4 \pm \sqrt{16 - 4 \cdot 13}}{2} = \frac{4 \pm \sqrt{-36}}{2} = \begin{cases} 2 + 3i \\ 2 - 3i \end{cases}$$

so the general solution is

$$y_h(t) = e^{2t}[c_1 \cos(3t) + c_2 \sin(3t)]$$

Using the first condition $y(0) = 1$, we get at once that $c_1 = 1$. After differentiating we have

$$\begin{aligned} y_h'(t) &= 2e^{2t}[\cos(3t) + c_2 \sin(3t)] + e^{2t}[-3\sin(3t) + 3c_2 \cos(3t)] \\ &= e^{2t}[(2 + 3c_2)\cos(3t) + (2c_2 - 3)\sin(t)] \end{aligned}$$

and using the second condition $y'(0) = 5$, we then have $2 + 3c_2 = 5$. Thus $c_2 = 1$, and our solution is

$$y(t) = e^{2t}[\cos(3t) + \sin(3t)]$$

Now, to get this to the form $y(t) = Ae^{2t}\cos(3t - \phi)$, we use the formulas

$$A = \sqrt{1^2 + 1^2} = \sqrt{2}$$

and

$$\phi = \arctan\left(\frac{1}{1}\right) = \frac{1}{4}\pi$$

which gives us

$$y(t) = \sqrt{2}e^{2t}\cos\left(3t - \frac{1}{4}\pi\right)$$

Chapter 3

Numerical solutions of ODEs

More often than not there exists no analytical solution of ODEs. Here the computer comes to our aid through numerical analysis. There are a great deal of resources available[1] to explore numerical solution methods of ODE's, so we are going to settle with looking at one methods. In fact, we will also restrict ourselves to the first order ODE's.

Let us start the short exposition by looking at what conditions need to be met for our ODE's to have a unique solution.

3.1 Existence and uniqueness

A reasonable question to ask is: How do we know for sure that the solution exists and is unique? We have two theorems for this, the first pertaining to just existence, and the second to both uniqueness and existence.

Theorem 3.1.1 Peano's Existence Theorem

Let $D \subset \mathbb{R}^2$ be an open set and $f : D \to \mathbb{R}$ be a continuous function. Then the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

with $(t_0, x_0) \in D$ has a solution on some open interval I containing x_0.

The solution need not be unique.

[1] One resource is for example https://folk.ntnu.no/leifh/teaching/tkt4140/

Definition 3.1.1 Lipschitz Continuity

f is Lipschitz continuous if there exists an $L > 0$ such that

$$|f(x) - f(y)| \leq L|x - y|,$$

for all $x, y \in I$.

Lipschitz continuity is a restriction on the growth of function. It states the slope of a function can never grow too large.

Theorem 3.1.2 Picard-Lindelöf Existence and Uniqueness Theorem

Suppose f be locally Lipschitz (Lipschitz on some open set containing 0), then there exists a unique solution of the initial value problem

$$\begin{cases} \dot{x}(t) = f(t, x(t)) \\ x(t_0) = x_0. \end{cases}$$

on some open set containing 0 (i.e. on a set $[0 - \epsilon, 0 + \epsilon]$ for some $\epsilon > 0$).

3.2 Euler's Method

Now that we have some criterion for the existence and uniqueness of a solution, let us start by exploring Euler's method of approximating such a solution.

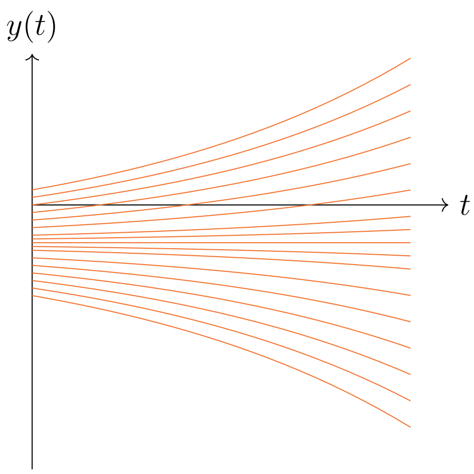


Figure 3.1: Solution curves of $\dot{y} - \frac{1}{4}y = \frac{1}{8}$

We consider an initial value problem

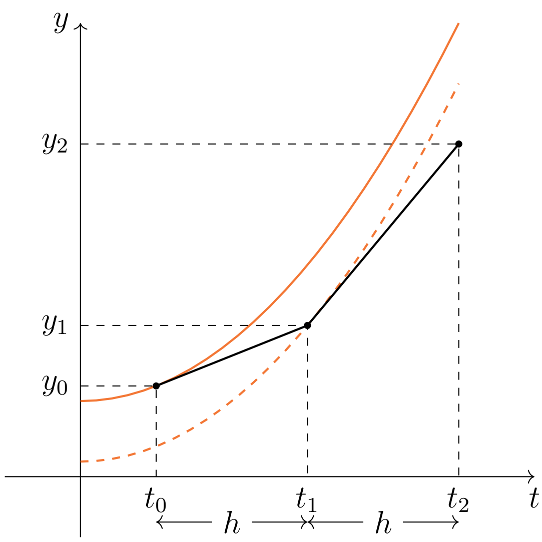
$$\begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}.$$

We assume f has sufficient assumptions (i.e. those of Picard-Linelöf) such that there is a unique solution to this problem. Through each point (t, y) in the ty-plane there is a possible solution curve of the differential equation.

A possible way to approximate one of these curves is the following: We start in a point (t_0, y_0) and move h steps along the tangent of the solution curve in that point. The point we are currently at is denoted (t_1, y_1). We now go h steps along the tangent of the solution curve going through that point and end up in (t_2, y_2). Once again we move h steps along the tangent of the solution curve in this point and end up in (t_3, y_3). When we iterate the same process further we are left with a list of points

$$(t_0, y_0), (t_1, y_1), (t_2, y_2), (t_3, y_3), \ldots, (t_n, y_n), \ldots$$

Drawing line segments between these points gives an approximation of the solution curve $y(t)$ of the initial value problem, and if we let the step size h move towards zero the approximation will converge to the solution.



Let us summarize the method.

Remark 3.2.1 Euler's method

When using Euler's method with a step size of h to approximate a solution of the initial value problem

$$\begin{cases} \dot{y}(t) = f(t, y(t)) \\ y(t_0) = y_0 \end{cases}.$$

we get a segmented line through the points (t_n, y_n) given iteratively as

$$t_n = t_0 + n \cdot h, \quad y_n = y_{n-1} + f(t_{n-1}, y_n - 1) \cdot h$$

with starting point (t_0, y_0).

Proof. Directly from the definition of the derivative we have,

$$\dot{y}(t_n) = \lim_{h \to 0} \left[\frac{y(t_n + h) - y(t_n)}{h} \right] \approx \frac{y(t_{n+1}) - y(t_n)}{h}.$$

We define the approximate value of the function at time step t_n as

$$y_n \approx y(t_n).$$

From this, using the fact that $\dot{y} = f(t, y)$, we define the method by

$$\frac{y_{n+1} - y_n}{h} = f(t_n, y_n)h,$$

or after rearranging

$$y_{n+1} = y_n + f(y_n).$$

□

Example 3.2.1

Let us use the Euler's method with step size $h = 1$ to approximate the solution to the initial value problem

$$\begin{cases} \dot{y} = f(t, y) = \frac{1}{8} + \frac{1}{4}y \\ y(0) = \frac{1}{2} \end{cases}$$

We have earlier calculated analytically that the solution to this problem is

$$y(t) = -\frac{1}{2} + e^{\frac{1}{4}t},$$

so we will also plot this and the approximated solution and see how close the approximation is. The initial condition tells us that our starting point (t_0, y_0)

is $(0, \frac{1}{2})$. Let us calculate the first four points.

$$\begin{aligned}
t_1 &= 1, & y_1 &= y_0 + f(t_0, y_0) \cdot h \\
& & &= \tfrac{1}{2} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{1}{2}\right) \cdot 1 \\
& & &= \tfrac{3}{4}
\end{aligned}$$

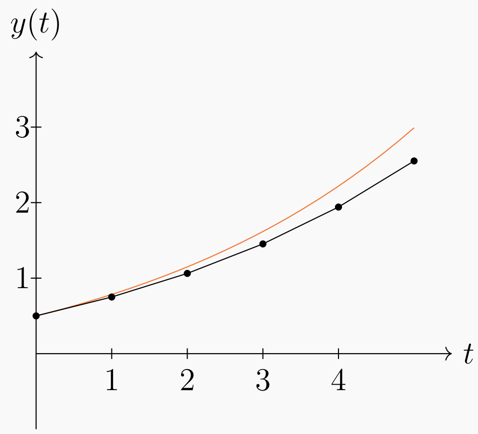
$$\begin{aligned}
t_2 &= 2, & y_2 &= y_1 + f(t_1, y_1) \cdot h \\
& & &= \tfrac{3}{4} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{3}{4}\right) \cdot 1 \\
& & &= \tfrac{17}{16}
\end{aligned}$$

$$\begin{aligned}
t_3 &= 3, & y_3 &= y_2 + f(t_2, y_2) \cdot h \\
& & &= \tfrac{17}{16} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{17}{16}\right) \cdot 1 \\
& & &= \tfrac{93}{64}
\end{aligned}$$

$$\begin{aligned}
t_4 &= 4, & y_4 &= y_3 + f(t_3, y_3) \cdot h \\
& & &= \tfrac{93}{64} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{93}{64}\right) \cdot 1 \\
& & &= \tfrac{497}{256}
\end{aligned}$$

$$\begin{aligned}
t_5 &= 5, & y_5 &= y_4 + f(t_4, y_4) \cdot h \\
& & &= \tfrac{497}{256} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{497}{256}\right) \cdot 1 \\
& & &= \tfrac{2613}{1024}
\end{aligned}$$

Now, let us plot these points and the exact analytical solution



Let us also try with step size $h = \frac{1}{2}$,

$$\begin{aligned} t_1 &= \tfrac{1}{2}, & y_1 &= y_0 + f(t_0, y_0) \cdot h \\ &&&= \tfrac{1}{2} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{1}{2}\right) \cdot \tfrac{1}{2} \\ &&&= \tfrac{5}{8} \end{aligned}$$

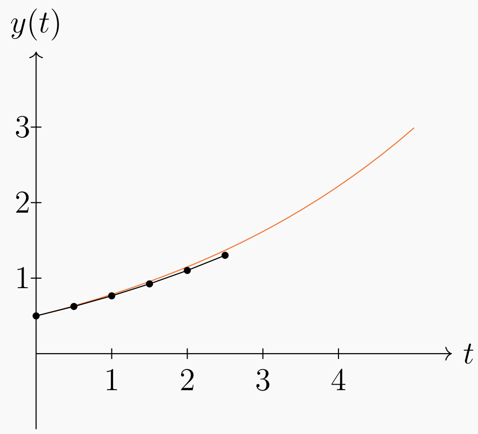
$$\begin{aligned} t_2 &= 1, & y_2 &= y_1 + f(t_1, y_1) \cdot h \\ &&&= \tfrac{5}{8} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{5}{8}\right) \cdot \tfrac{1}{2} \\ &&&= \tfrac{49}{64} \end{aligned}$$

$$\begin{aligned} t_3 &= 1.5, & y_3 &= y_2 + f(t_2, y_2) \cdot h \\ &&&= \tfrac{49}{64} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{49}{64}\right) \cdot \tfrac{1}{2} \\ &&&= \tfrac{473}{512} \end{aligned}$$

$$\begin{aligned} t_4 &= 2, & y_4 &= y_3 + f(t_3, y_3) \cdot h \\ &&&= \tfrac{473}{512} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{473}{512}\right) \cdot \tfrac{1}{2} \\ &&&= \tfrac{4513}{4096} \end{aligned}$$

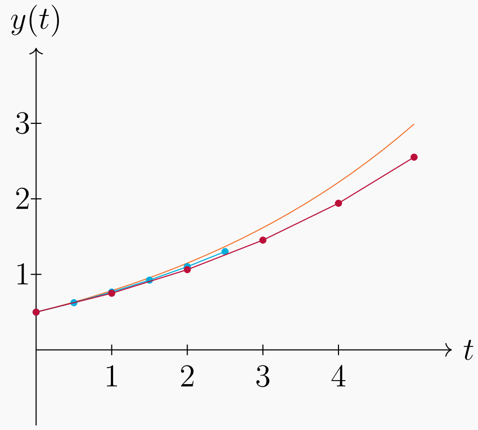
$$\begin{aligned} t_5 &= 2.5, & y_5 &= y_4 + f(t_4, y_4) \cdot h \\ &&&= \tfrac{4513}{4096} + \left(\tfrac{1}{8} + \tfrac{1}{4} \cdot \tfrac{4512}{4096}\right) \cdot \tfrac{1}{2} \\ &&&\approx 1.302032 \end{aligned}$$

Now, let us plot these points and the exact analytical solution.



And now let us plot both approximations, step size $h = 1$ in red and step size

$h = \frac{1}{2}$ in blue.



Remark 3.2.2

Observe on the last step in the example, we only get an approximate value for y_5. This is an example of the float point error one often gets when doing numerical analysis. The computer can only store a certain amount of decimals, so for every step after this we would get increasingly wrong estimations of y, on top of us already doing an approximation of the actual solution.

Part II

Linear Algebra

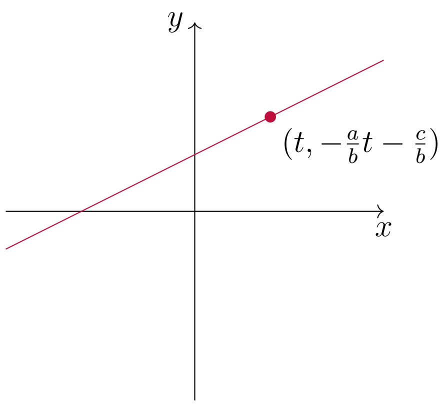
Chapter 4

Linear equations

A linear equation in one variable is on the form $ax + b = 0$ with a, b known values and x unknown. Such an equation has a unique solution $x = -b/a$. In two variables it is of the form $ax + by + c = 0$ with an infinite amount of solutions, namely the line

$$y = -\frac{a}{b}x - c$$

which can be visualized as a line in the xy-plane.



That is, for every number t we have a solution of the equation given by

$$x = t, \quad y = -\frac{a}{b}t - \frac{c}{b}.$$

In these cases we often call x a **free variable**. We could also have chosen y as a free variable, and then the solutions would have been on the form

$$x = -\frac{b}{a}r - \frac{c}{a}, \quad y = r$$

for every number r.

A linear equation in three variables is of the form $ax + by + cz + d = 0$ and the solutions will be points on a plane. Here we have two free variables, which we can choose to be y and z. Then the solutions will be on the form

$$x = -\frac{b}{a}t - \frac{c}{a}r - \frac{d}{a}, \quad y = t, \quad z = r$$

for every pair of numbers t and r.

Definition 4.0.1

A **linear equation** is of the form

$$a_1x_1 + a_2x_2 + \cdots + a_{n-1}x_{n-1} + a_nx_n = b$$

where $a_1, a_2, \ldots, a_{n-1}, a_n$ are known values, and $x_1, x_2, \ldots, x_{n-1}, x_n$ are unknown. The numbers a_i, for $i = 1, 2, \ldots, n$, are called the **coefficients** of the equation.

When you stumble upon linear equations in the wild you will more often than not have several equations describing the same unknowns x_i. That is, you have more than one constraint. This brings us to the concept of systems of linear equations.

Definition 4.0.2

A system of m linear equations in n variables is on the form of a collection of m linear equations with variables $x_1, x_2, \ldots, x_n$.

$$\begin{cases} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = b_2 \\ & & & & \vdots & & & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = b_m \end{cases}$$

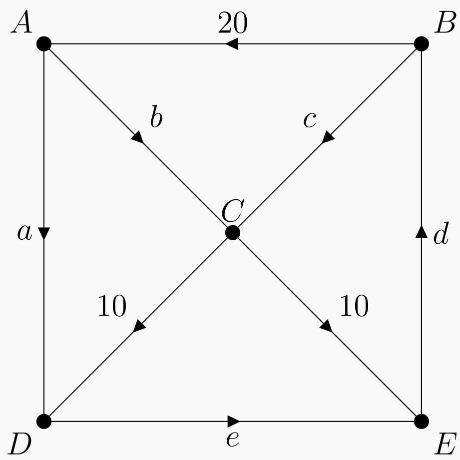
Example 4.0.1

When working with some quantity flowing through a network, say electric current, water or money, we can be interested in how much of the given quantity that flows through a specific part of the network.

A network consist of **junctions** or **nodes** with **branches** connecting them. Each branch has a given direction of flow and the underlying assumption is that the sum of everything flowing into a node should equal the amount flowing out.

This assumption is equal to Kirchoff's current law, when the network is a curcuit and the quantity is an electric current.

Let us look at the following network with some unknown quantities



The unknown quantities here are a, b, c, d. In order to find these, we use the assumption that what flows into the nodes A, B, C, D, E should equal what flows out of them to set up a system of linear equations.

$$\begin{aligned}
A: &\quad a + b = 20 \\
B: &\quad d - c = 20 \\
C: &\quad b + c = 20 \\
D: &\quad e - a = 10 \\
E: &\quad d - e = 10
\end{aligned}$$

Let us hold of on solving the system for a moment.

Example 4.0.2

Find the line $y = ax + b$ that intersect the points $(5, 8/2)$ and $(2, 5/2)$.

By substituting the values in the line equation, we obtain the following system of two linear equations with a and b unknown.

$$\begin{cases} 5a & + & b & = & \frac{8}{2} \\ 2a & + & b & = & \frac{5}{2} \end{cases}$$

Now we can either use substitution to solve this set, or we can add and subtract the equations. Let us try both methods.

Substitution: From the first equation we have that

$$b = \frac{8}{2} - 5a$$

substituting this value with b in the second equation gives us

$$2a + \frac{8}{2} - 5a = \frac{5}{2}$$

which in turn tells us that $a = \frac{1}{2}$. Now, by substituting this in our expression for b gives us $b = \frac{3}{2}$. That is the line is given by

$$y = \frac{1}{2}x + \frac{3}{2}$$

Adding and Subtracting equations: If we replace the second equation with the equation we get by subtracting the original first equation from the original second equation, we get

$$\begin{cases} 5a & + & b & = & \frac{8}{2} \\ -3a & + & 0 & = & -\frac{3}{2} \end{cases}$$

Now we can divide the second equation by -3:

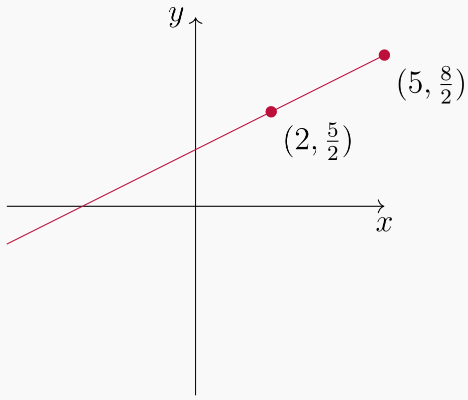
$$\begin{cases} 5a & + & b & = & \frac{8}{2} \\ a & + & 0 & = & \frac{1}{2} \end{cases}$$

before replacing the first equation, with the equation we get by subtracting five times the second equation from the first

$$\begin{cases} 0 & + & b & = & \frac{3}{2} \\ a & + & 0 & = & \frac{1}{2} \end{cases}$$

Reading out from this resulting system we get that the line is given by

$$y = \frac{1}{2} + \frac{3}{2}$$



4.1 Finding solutions

The last method used above is the one we are mostly interested in. It uses the same ideas as the method called Gaussian elimination, which is partly described through three elementary operations on the equations that do not change the solutions.

Remark 4.1.1

The three following elementary operations on systems do not change the resulting solutions.

1. Interchange any pair of equations
2. Multiply any equation by a nonzero number
3. Replace any equation by its sum with a multiple of any other equation.

Augmented matrix

When working with systems of linear equations there is a lot of symbols that can confuse us and obfuscate what we are doing. Hence we would like to simplify them and work only with the essential information.

Definition 4.1.1

The **coefficient matrix** of a system of linear equation

$$\begin{cases} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = b_2 \\ & & & & \vdots & & & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = b_m \end{cases}$$

is the array of numbers

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

consisting of m rows and n columns, where the columns contain every coefficient of the equations, such that

- column one has the coefficients of x_1,
  column two has the coefficient of x_2,
  ...
- row one has the coefficients from the first linear equation,
  row two has the coefficients from the second linear equation,
  ...

Let us also introduce the following **column vectors**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{bmatrix}$$

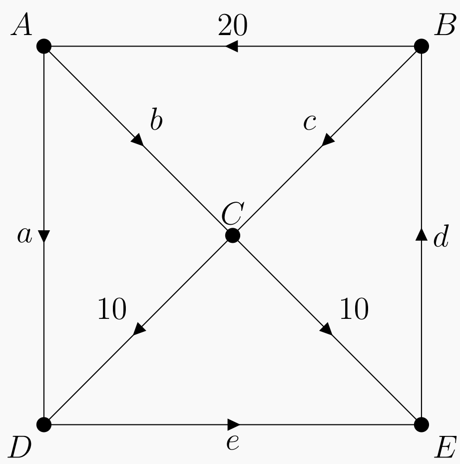
and define $A\mathbf{x}$ to be equal to

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{bmatrix}$$

We can see that if A is the coefficient matrix of a system of linear equations and $\mathbf{b}$ is the column vector defined above, then the **matrix equation** $A\mathbf{x} = \mathbf{b}$ encodes the same information as the system.

Example 4.1.1

Remember that the network



gave us the following system of linear equations

$$\begin{aligned} A:&\quad a+b=20\\ B:&\quad d-c=20\\ C:&\quad b+c=20\\ D:&\quad e-a=10\\ E:&\quad d-e=10 \end{aligned}$$

this in turn gives us

$$\overbrace{\begin{bmatrix} 1 & 1 & 0 & 0 & 0\\ 0 & 0 & -1 & 1 & 0\\ 0 & 1 & 1 & 0 & 0\\ -1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 1 & -1 \end{bmatrix}}^{A}\overbrace{\begin{bmatrix} a\\ b\\ c\\ d\\ e \end{bmatrix}}^{\mathbf{x}} = \overbrace{\begin{bmatrix} 20\\ 20\\ 20\\ 10\\ 10 \end{bmatrix}}^{\mathbf{b}}$$

Remark 4.1.2

If we have the matrix equation

$$\begin{bmatrix} a & b \end{bmatrix}\begin{bmatrix} x\\ y \end{bmatrix} = 0$$

and translate it to a linear equation $ax + by = 0$, we might recognize that the left hand side is nothing but the dot product of the two-dimensional vectors

$$\begin{bmatrix} a \\ b \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \end{bmatrix}.$$

Correspondingly for

$$\begin{bmatrix} a & b & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0$$

we see that the left hand side is equal to $ax + by + cz$ which is the dot product of the three-dimensional vectors

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \text{ and } \begin{bmatrix} x \\ y \\ z \end{bmatrix}.$$

In fact, we have that the **dot-product of n-dimensional vectors**

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

is defined to be exactly

$$\mathbf{x} \cdot \mathbf{y} = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} = x_1 y_1 + x_2 y_2 + \cdots + x_n y_n$$

Hence, we can see $A\mathbf{x}$ as being given as

$$\begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \vdots \\ \mathbf{a}_m \end{bmatrix} \mathbf{x} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{x} \\ \mathbf{a}_2 \cdot \mathbf{x} \\ \vdots \\ \mathbf{a}_m \cdot \mathbf{x} \end{bmatrix}$$

where $\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n$ are the rows of A seen as vectors.

When we are doing elementary operations on systems of linear equations, the only thing that changes in the corresponding matrix equation $A\mathbf{x} = \mathbf{b}$ are the rows in A and rows in $\mathbf{b}$. The changes in the matrix equation is equivalent to those on the system, and we can therefore work on the matrix equation instead of the linear system. Also, since $\mathbf{x}$ do not change, we can omit this in the calculation.

Definition 4.1.2

The augmented matrix of the system of linear equations

$$\begin{cases} a_{1,1}x_1 & + & a_{1,2}x_2 & + & \cdots & + & a_{1,n}x_n & = b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & \cdots & + & a_{2,n}x_n & = b_2 \\ & & & & \vdots & & & \\ a_{m,1}x_1 & + & a_{m,2}x_2 & + & \cdots & + & a_{m,n}x_n & = b_m \end{cases}$$

is given by

$$\left[\begin{array}{cccc|c} a_{1,1} & a_{1,2} & \cdots & a_{1,n} & b_1 \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} & b_m \end{array}\right]$$

The vertical line tells us what is on the right and left of the equality signs in the corresponding system. Now, the elementary operations can be formulated as

1. Interchange any pair of rows
2. Multiply any row by a nonzero number
3. Replace any row by its sum with a multiple of any other row.

Let us highlight this connection by going back to the equations in example 4.0.2.

Example 4.1.2

We were solving the system

$$\begin{cases} 5a & + & b & = & \frac{8}{2} \\ 2a & + & b & = & \frac{5}{2} \end{cases}$$

The augmented matrix of this system is

$$\left[\begin{array}{cc|c} 5 & 1 & \frac{8}{2} \\ 2 & 1 & \frac{5}{2} \end{array}\right]$$

$$\begin{cases} 5a & + & b & = & \frac{8}{2} \\ 2a & + & b & = & \frac{5}{2} \end{cases} \qquad \left[\begin{array}{cc|c} 5 & 1 & \frac{8}{2} \\ 2 & 1 & \frac{5}{2} \end{array}\right]$$

$$\xrightarrow{II=II-I} \begin{cases} 5a & + & b & = & \frac{8}{2} \\ -3a & + & 0 & = & -\frac{3}{2} \end{cases} \qquad \xrightarrow{II=II-I} \left[\begin{array}{cc|c} 5 & 1 & \frac{8}{2} \\ -3 & 0 & -\frac{3}{2} \end{array}\right]$$

$$\xrightarrow{II=\frac{1}{3}\cdot II} \begin{cases} 5a & + & b & = & \frac{8}{2} \\ a & + & 0 & = & \frac{1}{2} \end{cases} \qquad \xrightarrow{II=\frac{1}{3}\cdot II} \left[\begin{array}{cc|c} 5 & 1 & \frac{8}{2} \\ 1 & 0 & \frac{3}{2} \end{array}\right]$$

$$\xrightarrow{II=I-5\cdot II} \begin{cases} 0 & + & b & = & \frac{3}{2} \\ a & + & 0 & = & \frac{1}{2} \end{cases} \qquad \xrightarrow{I=I-5II} \left[\begin{array}{cc|c} 0 & 1 & \frac{3}{2} \\ 1 & 0 & \frac{1}{2} \end{array}\right]$$

Definition 4.1.3

A matrix is in **row echelon form** if the following is satisfied:

1. All nonzero rows are above any rows of all zeros.
2. Each leading entry of a row is in a column to the right of the leading entry of the row above it.
3. All entries in a column below a leading entry are zeros.

If a matris in echelon form also satisfies the following conditions, then it is in **reduced row echelon form**:

4. Each leading entry in each nonzero row is 1.
5. Each leading 1 is the only nonzero entry in its column.

The leading entries in the row echelon forms are called **pivot elements** and the columns they appear in are called **pivot columns**.

The following matrix is in row echelon form

$$\begin{bmatrix} 5 & 2 & 4 & 2 \\ 0 & 3 & 1 & 3 \\ 0 & 0 & 2 & 0 \end{bmatrix}$$

and the following matrix is in reduced row echelon form

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

If this had been the reduced row echelon form of the augmented matrix of a linear equation

$$\begin{array}{ccccccc} a_{1,1}x_1 & + & a_{1,2}x_2 & + & a_{1,3}x_3 & = & b_1 \\ a_{2,1}x_1 & + & a_{2,2}x_2 & + & a_{2,3}x_3 & = & b_2 \\ a_{3,1}x_1 & + & a_{3,2}x_2 & + & a_{3,3}x_3 & = & b_3 \end{array}$$

then we can read out from the reduced row echelon form

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{array}\right]$$

that the solution is given by $x_1 = 0$, $x_2 = 1$ and $x_3 = 0$.

Solution strategy

Remark 4.1.3

The general solution strategy for solving systems of linear equations is to

1. Find the augmented matrix of the system.
2. Use the elementary row operations to find the reduced row echelon form.
3. Read out the solution.

One can also in step two choose to only find the row echelon form and use substitution to find the other solutions.

Exactly one solution: If all the columns on the left of the vertical line contain a pivot element in the reduced row echelon form, then you have a unique solution to the system.

Infinite solutions: If you have columns on the left of the vertical line that contain non-zero entries, but are not pivot columns in the reduced row echelon form, then they correspond to **free variables** and you will have an infinite amount of solutions.

No Solutions: If you in the course of reduction get a row with only zero entries on the left of the vertical line, but non-zero entries on the right, then the system is called inconsistent and there will be no solution. This correspond to the equation $0 = 1$ which is ludicrous.

In example 4.0.2 we got a unique solution to the system. Now, let us look at the network in example 4.0.1 and observe that here we obtain an infinite amount of solutions.

Example 4.1.3 Infinite solutions

In example 4.0.1 we found the following matrix equation

$$\begin{bmatrix} 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} 20 \\ 20 \\ 20 \\ 10 \\ 10 \end{bmatrix}$$

we solve for (a, b, c, d, e) by reducing the following augmented matrix into reduced row echelon form

$$\left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ -1 & 0 & 0 & 0 & 1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array}\right] \xrightarrow{IV=IV+I} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array}\right]$$

$$\xrightarrow{II\leftrightarrow III} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 1 & 1 & 0 & 0 & 20 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array}\right] \xrightarrow{III=III-II} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & 1 & -10 \\ 0 & 0 & -1 & 1 & 0 & 20 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array}\right]$$

$$\xrightarrow{IV=IV+III} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 1 & -1 & 10 \end{array}\right] \xrightarrow{V=V-VI} \left[\begin{array}{ccccc|c} 1 & 1 & 0 & 0 & 0 & 20 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

$$\xrightarrow{I=I-II} \left[\begin{array}{ccccc|c} 1 & 0 & 0 & 0 & -1 & -10 \\ 0 & 1 & 0 & 0 & 1 & 30 \\ 0 & 0 & 1 & 0 & -1 & -10 \\ 0 & 0 & 0 & 1 & -1 & 10 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

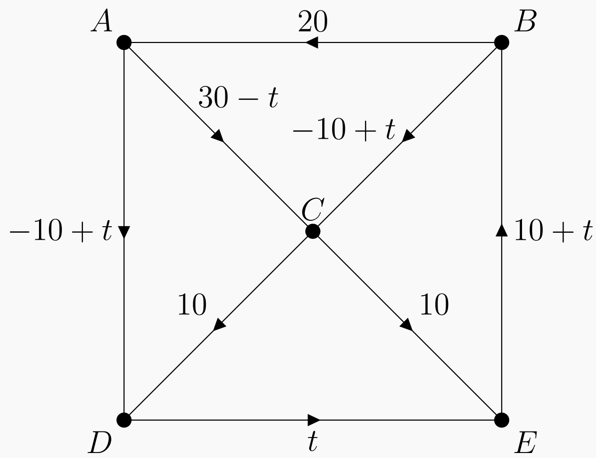
From the reduced form we read out that e is a free variable and the general solution of the system is

$$a = -10 + t, \ b = 30 - t, \ c = -10 + t, \ d = 10 + t, \ e = t$$

for any number t, or written in vector form

$$\begin{bmatrix} a \\ b \\ c \\ d \\ e \end{bmatrix} = \begin{bmatrix} -10+t \\ 30-t \\ -10+t \\ 10+t \\ t \end{bmatrix} = \begin{bmatrix} -10 \\ 30 \\ -10 \\ 10 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

where t times a vector means that we multiply each entry with t and addition of vectors is component wise.



The minus sign in the network tells us that the quantity moves in opposite direction of what is indicated by the arrow.

Example 4.1.4 No solutions

Let us try to solve the following system

$$\begin{cases} x - 2y = 1 \\ -5x + 10y = -1 \end{cases}$$

We reduce the augmented matrix

$$\left[\begin{array}{cc|c} 1 & -2 & 1 \\ -5 & 10 & -1 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & -2 & 1 \\ 0 & 0 & 4 \end{array}\right]$$

The last matrix correspond to the system

$$\begin{cases} x + 2y = 1 \\ 0x + 0y = 4 \end{cases}$$

The equation $0x + 0y = 4$ is equal to $0 = 4$, which can't be true, no matter the value we give x og y. The system has no solutions.

Let us look at a few more examples to warm up our thinking cap.

Example 4.1.5

We wish to solve the system

$$\begin{array}{ccccccc} 2x_1 & + & 3x_2 & + & x_3 & = & 16 \\ 3x_1 & + & x_2 & + & 4x_3 & = & 27 \\ 4x_1 & + & x_2 & + & x_3 & = & 18. \end{array}$$

Re-writing this as an augmented matrix, we have

$$\left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 3 & 1 & 4 & 27 \\ 4 & 1 & 1 & 18 \end{array}\right].$$

Performing the row operations,

$$\begin{aligned} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 3 & 1 & 4 & 27 \\ 4 & 1 & 1 & 18 \end{array}\right] & \xrightarrow[III=III-2\cdot I]{II=2\cdot II-3\cdot I} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 0 & -7 & 5 & 6 \\ 0 & -5 & -1 & -14 \end{array}\right] \\ & \xrightarrow{III=7\cdot III-5\cdot II} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 0 & -7 & 5 & 6 \\ 0 & 0 & -32 & -128 \end{array}\right] \\ & \xrightarrow{III=-\frac{1}{32}\cdot III} \left[\begin{array}{ccc|c} 2 & 3 & 1 & 16 \\ 0 & -7 & 5 & 6 \\ 0 & 0 & 1 & 4 \end{array}\right] \\ & \xrightarrow[II=II-5\cdot III]{I=I-III} \left[\begin{array}{ccc|c} 2 & 3 & 0 & 12 \\ 0 & -7 & 0 & -14 \\ 0 & 0 & 1 & 4 \end{array}\right] \end{aligned}$$

Rewriting this as a system, we have

$$\begin{array}{rcrcrcr} 2x_1 & + & 3x_2 & + & 0x_3 & = & 12 \\ 0x_1 & - & 7x_2 & + & 0x_3 & = & -14 \\ 0x_1 & + & 0x_2 & + & 1x_3 & = & 4. \end{array}$$

Thus $x_3 = 4$, $x_2 = 2$. Plugging these into the first equation, we have $x_1 = 3$. Written in vector form we have

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ 4 \end{bmatrix}$$

Example 4.1.6

We are given the following system to solve:

$$\begin{cases} x_1 + 3x_2 + 2x_3 + 3x_4 = 16 \\ x_1 + 3x_2 + 3x_3 + x_4 = 21 \\ 2x_1 + 6x_2 + 4x_3 + 6x_4 = 32 \end{cases}$$

As usual we make the augmented matrix and start reducing,

$$\left[\begin{array}{cccc|c} 1 & 3 & 2 & 3 & 16 \\ 1 & 3 & 3 & 1 & 21 \\ 2 & 6 & 4 & 6 & 32 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 2 & 3 & 16 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 3 & 0 & 7 & 6 \\ 0 & 0 & 1 & -2 & 5 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

The last matrix correspond to the following system

$$\begin{cases} x_1 + 3x_2 + 7x_4 = 6 \\ x_3 - 2x_4 = 5 \end{cases}$$

We have chosen to ignore the equation corresponding to the row of zeros. This is becuase it only tells us that $0x_1 + 0x_2 + 0x_3 + 0x_4 = 0$, or in other words $0 = 0$. This equation is satisfied regardless of the values for x_i, so it doesn't really tell us anything useful.

After moving everything except x_1 and x_3 to the right, we have:

$$\begin{cases} x_1 = -3x_2 - 7x_4 + 6 \\ x_3 = 2x_4 + 5 \end{cases}$$

That is, x_2 and x_4 are free variables. We can therefore find solutions of the system by setting x_2 and x_4 to be whatever we should please, and then using the equations to find what x_1 and x_3 needs to be.

For example, for $x_2 = 0$ and $x_4 = 1$, we get:

$$\begin{cases} x_1 = -3 \cdot 0 - 7 \cdot 1 + 6 = -1 \\ x_2 = 0 \\ x_3 = 2 \cdot 1 + 5 = 7 \\ x_4 = 1 \end{cases}$$

In order to describe all possible solutions, we set $x_2 = s$ and $x_4 = t$, for s and t arbitrary numbers. Then the general solution is:

$$\begin{cases} x_1 = -3s - 7t + 6 \\ x_2 = s \\ x_3 = 2t + 5 \\ x_4 = t \end{cases}$$

or written as vectors

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 6 \\ 0 \\ 5 \end{bmatrix} + s \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -7 \\ 0 \\ 2 \\ 1 \end{bmatrix}$$

Question 4.1.1

If we have a system of more variables than equation, and the system is solvable, will we always have free variables? Why?

Hint: We can at most have one pivot element in each row. Count rows and columns.

Some degree of freedom

When we are reducing a matrix, we have a certain degree of freedom. However, we can't do anything other than the three types of row operations. In what order and which rows we operate on are up to us though.

Example 4.1.7

We are reducing the following matrix:

$$\left[\begin{array}{cccc|c} 0 & 2 & 4 & 1 & 7 \\ 3 & 8 & 2 & 0 & 4 \\ 5 & 9 & 2 & 4 & 4 \end{array}\right]$$

Here we need to exchange the upper row with one of the other to get a pivot element in the right position. We can freely choose which row to move to the top.

We do also have some freedom when choosing the free variables. We can do as in example 4.1.6, that is to choose those variables which correspond to columns without pivot elements in the reduced form, but we can also choose other variables.

Example 4.1.8

In the example 4.1.6 we ended up with the two variables x_2 and x_4 being free. We could also have chosen x_1 and x_3 to be free.

We reduced the system to the following:

$$\begin{cases} x_1 = -3x_2 - 7x_4 + 6 \\ x_3 = 2x_4 + 5 \end{cases}$$

Solving the second equation for x_4:

$$x_4 = \frac{x_3 - 5}{2}$$

Substitution into the first equation gives:

$$\begin{aligned} x_2 &= \frac{-x_1 - 7x_4 + 6}{3} \\ &= \frac{-x_1 - \frac{7}{2}(x_3 - 5) + 6}{3} = -\frac{1}{3}x_1 - \frac{7}{6}x_3 + \frac{47}{6} \end{aligned}$$

Now, by setting $x_1 = s$ and $x_3 = t$, we obtain the following general solution:

$$\begin{cases} x_1 = s \\ x_2 = -\frac{1}{3}s - \frac{7}{6}t + \frac{47}{6} \\ x_3 = t \\ x_4 = \frac{1}{2}x_3 - \frac{5}{2} \end{cases}$$

Or in vector form:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{47}{6} \\ 0 \\ -\frac{5}{2} \end{bmatrix} + s \begin{bmatrix} 1 \\ -\frac{1}{3} \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ -\frac{7}{6} \\ 1 \\ \frac{1}{2} \end{bmatrix}$$

At first glance this looks different from what we had in 4.1.6, but it describes the same solutions. (If you let $s = -1$ and $t = 7$, we have the same solution as when we had $x_2 = 0$ and $x_4 = 1$ in example 4.1.6).

Remark 4.1.4

Do note that even if we have an infinite amount of solution, every variable **can't** necessarily be chosen as free. For example, if a system of three variables is reduced to

$$\begin{aligned} x + y &= 3 \\ 2z &= 8 \end{aligned}$$

Then either x or y can be chosen as being free, but z can **not** be free.

Chapter 5

Vectors and matrices

5.1 Vectors

We now introduce the most common vector, the ones used to model the real world. An abstract notion of a vector needs to be based off the properties that these vectors satisfy.

Definition 5.1.1 Vector in $\mathbb{R}^n$

Let n be some natural number, i.e. positive integer. Denote by $\mathbb{R}^n = \mathbb{R}\times\mathbb{R}\times\cdots\mathbb{R}$ the set of all n-tuples of real numbers. We write these in a few common notations. For $\mathbf{v} \in \mathbb{R}^n$,

- $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ (**column vector notation**),
- $\mathbf{v} = \begin{bmatrix} v_1 & v_2 & \cdots & v_n \end{bmatrix}$ (**row vector notation**),
- $\mathbf{v} = (v_1, v_2, \dots, v_n)$ (**coordinate notation**).

Vectors are usually denoted by a bold lowercase letter, $\mathbf{v}$ in typed texts. If written by hand it is common to denote vectors by an arrow above, $\vec{v}$, or a line below, $\underline{v}$, instead of boldtype.

Remark 5.1.1

Unless explicitly told otherwise we will by a vector mean a column vector. Also, to save vertical space we will often write out column vectors as list of numbers. That is $\mathbf{v} = (1, 2, 3)$ will be the same as

$$\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

We wish to explore some properties of vectors. Let $\mathbf{u}, \mathbf{v} \in \mathbb{R}^n$ and $\alpha \in \mathbb{R}$. Then the following properties hold

- $\alpha \mathbf{v} = \alpha \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} \alpha v_1 \\ \alpha v_2 \\ \vdots \\ \alpha v_n \end{bmatrix}$ (**scalar multiplication**),
- $\mathbf{u} + \mathbf{v} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \\ \vdots \\ u_n + v_n \end{bmatrix}$ (**vector addition**).

We also have some geometric properties of vectors. Note that vectors have a sense of direction and length,

- the **dot product/inner product** of two vectors is given by $\langle \mathbf{u}, \mathbf{v} \rangle = \mathbf{u} \cdot \mathbf{v} = u_1 v_2 + u_2 v_2 + \ldots u_n v_n$,
- The **magnitude/length** of a vector is given by $|v| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2} = \sqrt{\mathbf{v} \cdot \mathbf{v}}$,
- the angle between two vectors, θ is given by the relation $\mathbf{u} \cdot \mathbf{v} = |u||v| \cos(\theta)$,
- two vectors are orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$, or if the angle between them is a right angle.

A vector on the form

$$a\mathbf{x} + b\mathbf{y} = a \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_m \end{bmatrix} + b \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} ax_1 + by_1 \\ ax_2 + by_2 \\ \vdots \\ ax_m + by_m \end{bmatrix}$$

is a **linear combination** of the vectors $\mathbf{x}$ and $\mathbf{y}$. The numbers a and b is called scalars. If we have n vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, then a vector on the form

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n$$

is a linear combination of the vectors $\mathbf{x}_i$ with scalars a_1.

If we have n vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$, we define the linear span, or

$$\mathrm{span}(\mathbf{x}_1, \ldots, \mathbf{x}_n)$$

to be the set of all linear combinations of $\mathbf{x}_1, \ldots, \mathbf{x}_n$, that is every possible scaled sum

$$a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_n\mathbf{x}_n$$

Example 5.1.1

The vector $\begin{bmatrix} 11 \\ 16 \\ 21 \end{bmatrix}$ is a linear combination of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ og $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$. since

$$3\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + 2\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix} = \begin{bmatrix} 11 \\ 16 \\ 21 \end{bmatrix}$$

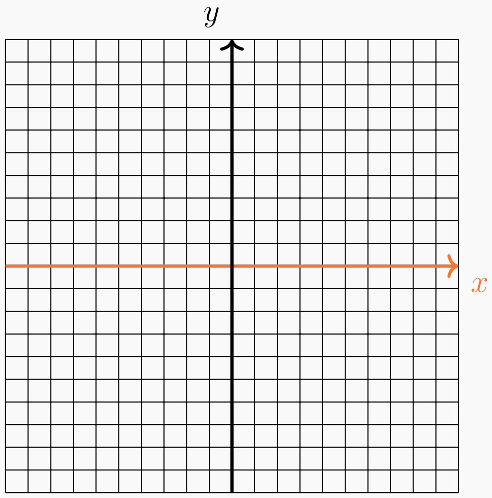
The span of $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ og $\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ are all vectors i $\mathbb{R}^3$ on the form

$$a\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + b\begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix},$$

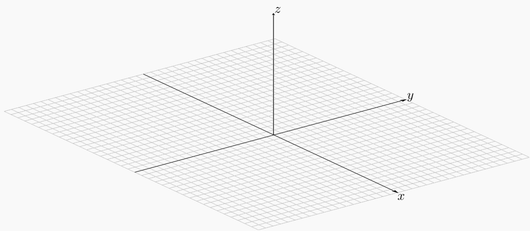
with a and b numbers.

In $\mathbb{R}^2$ and $\mathbb{R}^3$ such sets can be easily visualized.

Example 5.1.2



a) The span of the set $\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ is the x-axis in the xy-plane.



b) The span of the set $\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^3$ (or the xyz-space) are all vectors on the form

$$\begin{bmatrix} a \\ b \\ 0 \end{bmatrix} = a \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

for all values of a and b. This is the xy-plane.

Vector equations

Let us look back to the definitions leading up to augmented matrices, specifically the matrix equation $A\mathbf{x} = \mathbf{b}$. The left hand side of the equation was defined to be

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{bmatrix}$$

and this can be split into a scaled sum of the column vectors of A

$$\begin{bmatrix} a_{1,1}x_1 + a_{1,2}x_2 + \cdots + a_{1,n}x_n \\ a_{2,1}x_1 + a_{2,2}x_2 + \cdots + a_{2,n}x_n \\ \vdots \\ a_{m,1}x_1 + a_{m,2}x_2 + \cdots + a_{m,n}x_n \end{bmatrix} = x_1 \begin{bmatrix} a_{1,1} \\ a_{2,1} \\ \vdots \\ a_{m,1} \end{bmatrix} + x_2 \begin{bmatrix} a_{1,2} \\ a_{2,2} \\ \vdots \\ a_{m,2} \end{bmatrix} + \cdots + x_n \begin{bmatrix} a_{1,n} \\ a_{2,n} \\ \vdots \\ a_{m,n} \end{bmatrix}.$$

Remark 5.1.2

Observe that $A\mathbf{x}$ is the sum of the column vectors of A, scaled by the entries of $\mathbf{x}$, i.e.

$$A\mathbf{x} = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \cdots + x_n\mathbf{a}_n$$

Thus $A\mathbf{x} = \mathbf{b}$ can also be seen as a **vector equation**.

This gives us a new way to look at systems of linear equations: The problem is to find the scales x_i such that the linear combination of the matrix columns equals the right hand side.

Example 5.1.3

The vector equation

$$x \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} + y \begin{bmatrix} 2 \\ 5 \\ 5 \end{bmatrix} + z \begin{bmatrix} -2 \\ 9 \\ -1 \end{bmatrix} = \begin{bmatrix} -5 \\ 33 \\ 0 \end{bmatrix}.$$

is equivalent to the system of equation

$$\begin{cases} x + 2y - 2z = -5 \\ x + 5y + 9z = 33 \\ 2x + 5y - z = 0 \end{cases}$$

and has a unique solution:

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 7 \\ -2 \\ 4 \end{bmatrix},$$

Verify yourselves that the following is in fact an equality

$$7\begin{bmatrix}1\\1\\2\end{bmatrix} - 2\begin{bmatrix}2\\5\\5\end{bmatrix} + 4\begin{bmatrix}-2\\9\\-1\end{bmatrix} = \begin{bmatrix}-5\\33\\0\end{bmatrix}.$$

Remark 5.1.3

If the right hand side of a matrix equation $A\mathbf{x} = \mathbf{b}$ is zero, i.e. $\mathbf{b} = \mathbf{0}$, then the equation is called **homogeneous**, and if it is nonzero, $\mathbf{b} \neq \mathbf{0}$, then the equation is called **non-homogeneous**.

The general solution of a matrix equation $A\mathbf{x} = \mathbf{b}$ is given by the sum of the general solution of the homogeneous case $\mathbf{x}_h$ and a particular solution of the non-homogeneous case $\mathbf{x}_p$,

$$\mathbf{x} = \mathbf{x}_h + \mathbf{x}_p.$$

Linear Transformations

When working with vectors we will often use a particular type of function on them that behaves nicely. Let us indulge in a small description of these.

Definition 5.1.2 Linear Transformation

We call a function $T : \mathbb{R}^n \to \mathbb{R}^m$ a **linear transformation** if it preserves scalar multiplication and vector addition, that is

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y),$$

for any $\alpha, \beta \in \mathbb{R}$ and $x, y \in \mathbb{R}^n$.

Example 5.1.4 Some linear transformations

The following are some examples of linear transformations:

- $\mathrm{Id} : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}x_1\\x_2\end{bmatrix}$ (**Identity Mapping**),
- $S : \mathbb{R}^2 \to \mathbb{R}^2, \begin{bmatrix}x_1\\x_2\end{bmatrix} \mapsto \begin{bmatrix}x_2\\x_1\end{bmatrix}$,
- $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x_1, x_2, x_3) = (x_1, 2x_2, x_3)$,

- $P : \mathbb{R}^3 \to \mathbb{R}^2, P(x_1, x_2, x_3) = (x_1, x_2)$.

You can confirm these are linear transformations directly using the definition.

We also introduce an examples of a mapping that is **not** a linear transformation.

- $\tilde{T} : \mathbb{R}^2 \to \mathbb{R}^2, (x_1, x_2) \mapsto (x_1^2, x_2)$.

Linear transformations are particularly nice functions, however in the forms given they are not necessarily nice to work with. Is there a way we can write these in a nice way?

5.2 Matrices

We have for some time now worked with matrices without really going into the gritty details. Let us look a bit more closely on what they are. We will se that vectors are a special type of matrices, and that the operations possible on them extends nicely to matrices as well.

Definition 5.2.1

A **matrix** is a rectangular array of numbers. A matrix A having m rows and n columns is written as

$$A = \begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}.$$

The **size** of a matrix is the number of rows and columns, and we say that a matrix A is of size $m \times n$, or A is an $m \times n$-matrix, if it has m rows and n columns. Two matrices of equal size and with the same elements in corresponding positions are equal.

Remark 5.2.1

Another useful notion when working with matrices is to consider the matrix being built up of column vectors. If we let $\mathbf{a}_j$ be the column vector consisting of the elements of the jth column of a $m \times n$-matrix A, then we can write A as

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a_2} & \cdots & \mathbf{a_n} \end{bmatrix}$$

$$\begin{array}{cc} & \begin{array}{ccccc} & & \text{Column} & & \\ & & j & & \end{array} \\ \begin{array}{c} \text{Row} \\ i \end{array} & \begin{bmatrix} a_{1,1} & \cdots & a_{1,j} & \cdots & a_{1,n} \\ \vdots & & \vdots & & \vdots \\ a_{i,1} & \cdots & a_{i,j} & \cdots & a_{i,n} \\ \vdots & & \vdots & & \vdots \\ a_{m,1} & \cdots & a_{m,j} & \cdots & a_{m,n} \end{bmatrix} \\ & \begin{array}{ccccc} \uparrow & & \uparrow & & \uparrow \\ \mathbf{a}_1 & & \mathbf{a}_j & & \mathbf{a}_n \end{array} \end{array}$$

Example 5.2.1 Nice 2×2-matrices

Identity matrix:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Zero matrix:

$$\mathbf{0} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Symmetric and **anti-symmetric** matrices:

$$\begin{bmatrix} a & b \\ b & c \end{bmatrix}, \begin{bmatrix} 0 & d \\ -d & 0 \end{bmatrix}$$

for numbers a, b, c, d.

Upper- and lower triangular matrices:

$$\begin{bmatrix} a & b \\ 0 & c \end{bmatrix}, \begin{bmatrix} d & 0 \\ e & f \end{bmatrix}$$

for numbers a, b, c, d, e, f.

Question 5.2.1

Can you write down 3×3 (anti-)symmetric and upper/lower triangular matrices?

Let us look at some uglier matrices as well.

Example 5.2.2

$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \end{bmatrix}$	2 rows and 3 columns	(2×3)-matrix
$B = \begin{bmatrix} 3 & 8 \\ 9 & 8 \\ 0 & 1 \end{bmatrix}$	3 rows and 2 columns	(3×2)-matrix
$C = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix}$	3 rows and 3 columns	(3×3)-matrix
$D = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$	3 rows and 1 columns	(3×1)-matrix
$E = \begin{bmatrix} 1 & 3 & 2 \end{bmatrix}$	1 rows and 3 columns	(1×3)-matrix

Note that the matrix D and the matrix E can also be seen as vectors. Hence we give the following alternative definition of vectors.

Definition 5.2.2 Alternative vector definition

A **vector** is a matrix having either only one row or one column. Matrices having only one column ($n \times 1$-matrices) are called **column vectors** and matrices having only one row ($1 \times n$) are called **row vectors**.

Theorem 5.2.1

Let A be an $m \times n$-matrix, $\alpha, \beta \in \mathbb{R}$ a number and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ vectors, then

$$A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$$

Equivalently, the function $T_A \colon \mathbb{R}^n \to \mathbb{R}^m$ given by

$$T_A(\mathbf{x}) = A\mathbf{x}$$

is a linear transformation.

That is, **Any mapping defined by a matrix is a linear transformation.**

Actually, we have that the linear transformations described this way gives us every linear transformation.

Theorem 5.2.2

Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. There exists a unique $m \times n$-matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}, \qquad (\mathbf{x} \in \mathbb{R}^n).$$

This matrix is given by

$$A = \begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & \cdots & T(\mathbf{e}_n) \end{bmatrix}$$

where $\mathbf{e}_i$ is **the ith standard basis vector** of $\mathbf{R}^n$, which consists of a 1 in the ith row and zero elsewhere.

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

Example 5.2.3 Matrices for some Linear Transformations

Consider the linear transformations from Example 5.1.4. We will rewrite these using a matrix.

- Id : $\mathbb{R}^2 \to \mathbb{R}^2$ has the matrix $I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$,
- $S : \mathbb{R}^2 \to \mathbb{R}^2, S(x) = Ax$ has the matrix $A = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$,
- $T : \mathbb{R}^3 \to \mathbb{R}^3, T(x) = bx$ has the matrix $B = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.

5.3 Matrix operations

Addition and subtraction

As with numbers, we can add and subtract matrices. This is done element wise and only if the matrices are of the same size.

Example 5.3.1

$$\begin{bmatrix} 1 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 2 & 5 & 1 \\ 9 & 3 & 0 \end{bmatrix} = \begin{bmatrix} 1+2 & 2+5 & 3+1 \\ 7+9 & 3+3 & 2+0 \end{bmatrix} = \begin{bmatrix} 3 & 7 & 4 \\ 16 & 6 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 3 \\ 7 & 1 \\ 5 & 9 \end{bmatrix} - \begin{bmatrix} 2 & 8 \\ 3 & 1 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} -2 & -5 \\ 4 & 0 \\ 5 & 6 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 7 & 1 \\ 5 & 9 \end{bmatrix} = \text{undefined}$$

Addition on numbers is commutative, meaning that the order of the numbers are indifferent.

$$34 + 79 = 113$$
$$79 + 34 = 113$$

There is also a special number (additive identity) under addition which has no effect on the other, namely *zero*.

$$340 + 0 = 340$$

$$0 + 59 = 59$$

Both of these properties have their analogue for addition in matrices, since addition on those are defined element wise.

Theorem 5.3.1

Let A, B and C be matrices of equal size $m \times n$, then

- $A + B = B + A$ (Additive commutativity)
- $(A + B) + C = A + (B + C)$ (Additive associativity)
- $A + \mathbf{0} = A$ (Additive identity)

where $\mathbf{0}$ is the zero matrix of size $m \times n$, that is the matrix with all elements equal zero.

Scalar multiplication

If we have a number c and a matrix A, then we can *scale* the matrix using the operation **scalar multiplication**. This is done by taking the produkt of c element wise on the matrix A, and we denote the resulting matrix with cA.

$$cA = c\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix} = \begin{bmatrix} ca_{1,1} & ca_{1,2} & \cdots & ca_{1,n} \\ ca_{2,1} & ca_{2,2} & \cdots & ca_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ ca_{m,1} & ca_{m,2} & \cdots & ca_{m,n} \end{bmatrix}$$

Example 5.3.2

Let us look at the scalar multiplication of the matrices from example 5.2.2

$$3A = 3\begin{bmatrix} 1 & 2 & 3 \\ 0 & 3 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 6 & 9 \\ 0 & 9 & 3 \end{bmatrix}$$

$$4B = 4\begin{bmatrix} 3 & 8 \\ 9 & 8 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 32 \\ 36 & 32 \\ 0 & 4 \end{bmatrix}$$

$$2C = 2\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 0 \\ 4 & 6 & 2 \\ 0 & 8 & 2 \end{bmatrix}$$

$$5D = 5\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \\ 20 \end{bmatrix}$$

$$8E = 8\begin{bmatrix} 1 & 3 & 2 \end{bmatrix} = \begin{bmatrix} 8 & 24 & 16 \end{bmatrix}$$

This operation also inherit some good looking properties from multiplication of numbers. Recall that when you multiply numbers with an addition of numbers in parentheses you can either sum up the addition and then multiply or multiply each summand and then add.

$$3(4+2) = \begin{cases} 3 \cdot 6 = 18 \\ 3 \cdot 4 + 3 \cdot 2 = 12 + 6 = 18 \end{cases}$$

$$(5+1)4 = \begin{cases} 6 \cdot 4 = 24 \\ 5 \cdot 4 + 1 \cdot 4 = 20 + 4 = 24 \end{cases}$$

Theorem 5.3.2

Let A and B be matrices of equal size $m \times n$ and c, d be two numbers, then

- $c(dA) = (cd)A$

- $c(A + B) = cA + cB$
- $(c + d)A = cA + dA$
- $1A = A$

Multiplication of matrices

Another operation of numbers which we would like to emulate for matrices are multiplication. There is a naivé way of defining such an operation by doing it element wise. This turns out to be of little to no benefit, so let us disregard that idea at once.

Let us recall that linear transformations are uniquely determined by a matrix. Therefore, when multiplying a vector $\mathbf{x} \in \mathbb{R}^n$ with a $m \times n$-matrix, we can think of it as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. We want to think of multiplication of two matrices as the composition of two function, and therefore the following equality has to hold:

$$(AB)\mathbf{x} = A(B\mathbf{x})$$

The linear transformation T_B (multiplication by B) acts first and then the transformation T_A (multiplication by A). If A is a $m \times n$-matrix and B a $q \times p$-matrix, then we need $n = p$ for this to make sense, since T_B sends vectors in $\mathbb{R}^p$ to vectors in $\mathbb{R}^q$ and T_A sends vectors in $\mathbb{R}^n$ to vectors in $\mathbb{R}^m$.

$$\begin{tikzcd} \mathbb{R}^q \arrow[r, "T_B"] \arrow[rr, bend right, "T_{AB}"'] & \mathbb{R}^p = \mathbb{R}^n \arrow[r, "T_A"] & \mathbb{R}^m \end{tikzcd}$$

The composition of T_B with T_A will be T_{AB}. Before defining what AB has to be for this to make sense, let us look at an example.

Example 5.3.3

Let A and B be the following two 2×2-matrices:

$$A = \begin{bmatrix} 3 & -5 \\ 2 & 7 \end{bmatrix} \qquad B = \begin{bmatrix} 4 & 3 \\ 2 & 1 \end{bmatrix}$$

We want to find the matrix AB such that $(AB)\mathbf{v} = A(B\mathbf{v})$ for all vectors $\mathbf{v}$ i $\mathbb{R}^2$.

We first look at multiplication with the standard basis vectors

$$\begin{bmatrix}1\\0\end{bmatrix} \quad \text{og} \quad \begin{bmatrix}0\\1\end{bmatrix}.$$

We know from the definition of matrix multiplication on vectors that by multiplying a 2×2-matrix with one of these vectors, we obtain either the first or second column of the matrix.

$$B\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}4\\2\end{bmatrix}$$
$$A\left(B\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}3 & -5\\2 & 7\end{bmatrix}\begin{bmatrix}4\\2\end{bmatrix} = \begin{bmatrix}2\\22\end{bmatrix}$$

We want AB to be given in a way that satisfies

$$(AB)\mathbf{v} = A(B\mathbf{v})$$

for all vectors $\mathbf{v}$. In particular:

$$(AB)\begin{bmatrix}1\\0\end{bmatrix} = A\left(B\begin{bmatrix}1\\0\end{bmatrix}\right) = \begin{bmatrix}2\\22\end{bmatrix}$$

Since multiplication with

$$\begin{bmatrix}1\\0\end{bmatrix}$$

gives us the first column of the matrix, we know that the first column of AB has to be

$$\begin{bmatrix}2\\22\end{bmatrix}$$

Likewise the second of column of AB is:

$$B\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}3\\1\end{bmatrix}$$
$$A\left(B\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}3 & -5\\2 & 7\end{bmatrix}\begin{bmatrix}3\\1\end{bmatrix} = \begin{bmatrix}4\\13\end{bmatrix}$$
$$(AB)\begin{bmatrix}0\\1\end{bmatrix} = A\left(B\begin{bmatrix}0\\1\end{bmatrix}\right) = \begin{bmatrix}4\\13\end{bmatrix}$$

Then we see that the second column of AB needs to be:

$$\begin{bmatrix} 4 \\ 13 \end{bmatrix}$$

So the product of A og B is:

$$AB = \begin{bmatrix} 2 & 4 \\ 22 & 13 \end{bmatrix}$$

In general if A is of size $m \times n$, B is of size $n \times p$ and $\mathbf{x} \in \mathbb{R}^p$, then

$$B\mathbf{x} = x_1\mathbf{b}_1 + x_2\mathbf{b}_2 + \cdots + x_p\mathbf{b}_p$$

for $\mathbf{b}_1, \ldots, \mathbf{b}_p$ the columns of B and $x_1, \ldots, x_n$ the elements of $\mathbf{x}$. Now using that $A(\alpha\mathbf{x} + \beta\mathbf{y}) = \alpha A\mathbf{x} + \beta A\mathbf{y}$, we have

$$A(B\mathbf{x}) = x_1 A\mathbf{b}_1 + x_2 A\mathbf{x}_2 + \cdots + x_p A\mathbf{b}_p$$

We recognize this as being equal to

$$\begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix} \mathbf{x}$$

and we conclude that $AB = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 & \cdots & A\mathbf{b}_p \end{bmatrix}$, or using the definition of multiplication of matrix with vector, we have

$$AB = \begin{bmatrix} \mathbf{a}_1\mathbf{b}_1 & \mathbf{a}_1\mathbf{b}_2 & \cdots & \mathbf{a}_1\mathbf{b}_p \\ \mathbf{a}_2\mathbf{b}_1 & \mathbf{a}_2\mathbf{b}_2 & \cdots & \mathbf{a}_2\mathbf{b}_p \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{a}_m\mathbf{b}_1 & \mathbf{a}_m\mathbf{b}_2 & \cdots & \mathbf{a}_m\mathbf{b}_p \end{bmatrix}$$

where $\mathbf{a}_1, \ldots \mathbf{a}_n$ is the row vectors of A.

Definition 5.3.1

Let A be an $m \times n$-matrix and B an $n \times p$-matrix. Their product AB is an $m \times p$ matrix where the element $(AB)_{i,j}$ is

$$(AB)_{i,j} = a_{i,1}b_{1,j} + a_{i,2}b_{2,j} + \cdots + a_{i,n}b_{n,j} = \sum_{k=1}^{n} a_{i,k}b_{k,j}$$

which is the same as saying that $(AB)_{i,j}$ is the dot product of the ith row vector of A and the jth column vector of B.

$$
\underset{\text{row } i}{m\left.\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ \vdots & \vdots & & \vdots \\ a_{i,1} & a_{i,2} & \cdots & a_{i,n} \\ \vdots & \vdots & & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}\right.}^{\textstyle n}
\underset{\times}{\times}
\underset{\text{column } j}{n\left.\begin{bmatrix} b_{1,1} & \cdots & b_{1,j} & \cdots & b_{1,p} \\ b_{2,1} & \cdots & b_{2,j} & \cdots & b_{2,p} \\ \vdots & & \vdots & & \vdots \\ b_{n,1} & \cdots & b_{n,j} & \cdots & b_{n,p} \end{bmatrix}\right.}^{\textstyle p}
\underset{=}{=}
\underset{(i,j)\text{-element}}{m\left.\begin{bmatrix} ab_{1,1} & \cdots & ab_{1,j} & \cdots & ab_{1,p} \\ \vdots & & \vdots & & \vdots \\ ab_{i,1} & \cdots & ab_{i,j} & \cdots & ab_{i,p} \\ \vdots & & \vdots & & \vdots \\ ab_{m,1} & \cdots & ab_{m,j} & \cdots & ab_{m,p} \end{bmatrix}\right.}^{\textstyle p}
$$

Example 5.3.4

A possibly easier way of visualizing the process is the following.

Say we want to calculate the following product

$$
\overbrace{\begin{bmatrix} 1 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix}}^{2\times 3}
\overbrace{\begin{bmatrix} 0 & 3 \\ 7 & 1 \\ 5 & 9 \end{bmatrix}}^{3\times 2}
=
\overbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}^{2\times 2}
$$

then we can organize the matrices as following

$$
\begin{array}{ccc|cc}
 & & & 0 & 3 \\
 & & & 7 & 1 \\
 & & & 5 & 9 \\
\hline
1 & 2 & 3 & a & b \\
7 & 3 & 2 & c & d
\end{array}
$$

The second matrix is placed in the upper right corner and the first matrix in the lower left corner. In order to find the elements in the product, we take the dot product of the row and column pointing in to the element.

$$
\begin{array}{ccc|cc}
 & & & 0 & 3 \\
 & & & 7 & 1 \\
 & & & 5 & 9 \\
\hline
1 & 2 & 3 & a & b \\
7 & 3 & 2 & c & d
\end{array}
$$

$$a = 1 \cdot 0 + 2 \cdot 7 + 3 \cdot 5 = 29$$

$$
\begin{array}{ccc|cc}
 & & & 0 & 3 \\
 & & & 7 & 1 \\
 & & & 5 & 9 \\
\hline
1 & 2 & 3 & a & b \\
7 & 3 & 2 & c & d
\end{array}
$$

$$b = 1 \cdot 3 + 2 \cdot 1 + 3 \cdot 9 = 32$$

This gives us

$$
\begin{bmatrix} 1 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix}
\begin{bmatrix} 0 & 3 \\ 7 & 1 \\ 5 & 9 \end{bmatrix}
=
\begin{bmatrix} 29 & 32 \\ 31 & 42 \end{bmatrix}
$$

Remark 5.3.1

Be aware that matrix multiplication is only defined if the matrices has the right corresponding sizes.

$$[m \times \cancel{n}] \cdot [\cancel{n} \times p] = m \times p$$

Example 5.3.5

a) Calculate, if possible,

$$
\begin{bmatrix} 1 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix}
\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix}
$$

$$\begin{array}{ccc|ccc} & & & 1 & 1 & 0 \\ & & & 2 & 3 & 1 \\ & & & 0 & 4 & 1 \\ \hline 1 & 2 & 3 & 5 & 19 & 5 \\ 7 & 3 & 2 & 13 & 24 & 5 \end{array}$$

$$\begin{bmatrix} 1 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \underline{\begin{bmatrix} 5 & 19 & 5 \\ 13 & 24 & 5 \end{bmatrix}}$$

b) Calculate, if possible,

$$\begin{bmatrix} 1 & 1 & 0 \\ 2 & 3 & 1 \\ 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 7 & 3 & 2 \end{bmatrix}$$

This is undefined since the number of columns in the first matrix do not equal the number of rows in the second matrix.

We have seen that additive properties of numbers have corresponding properties for addition of matrices. The same holds true for multiplication, except for one crucial detail, namely commutativity, i.e. $a \times b = b \times a$. We have already seen an extreme way it fails to commute in the preceding example, namely because of size constraints, but as we will now see it is generally not true even for square matrices $(n \times n)$.

Example 5.3.6 Non-commutivity

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

However, we have a lot of other nice properties of matrix-multiplication.

Theorem 5.3.3

Let A, B and C be matrices such that the operations described below is defined, and d a number. Then

- Multiplicative associativity

$$A(BC) = (AB)C$$

- Multiplicative distributivity

$$A(B+C) = AB + AC \text{ and } (A+B)C = AC + BC$$

- $(dA)B = d(AB) = A(dB)$

Let now I_n denote a $n \times n$-matrix with 1's on the diagonal and 0's everywhere else, then for A a $m \times n$-matrix we have

- $AI_n = A = I_m A$. Thus I_n is called the $n \times n$ identity matrix.

Remark 5.3.2

Let us stress the fact that for two matrices A and B, we do in general ***not*** have

$$AB = BA$$

Example 5.3.7

Let A, B and C be the following matrices:

$$A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 1 \\ 1 & 0 \\ 2 & 4 \end{bmatrix} \qquad C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

A is of size 2×3 and B is of size 3×2, so AB must be of size 2×2. Let us calculate this using the definition:

$$\begin{aligned} AB &= \begin{bmatrix} 5 \cdot 2 + 0 \cdot 1 + (-2) \cdot 2 & 5 \cdot 1 + 0 \cdot 0 + (-2) \cdot 4 \\ 3 \cdot 2 + 1 \cdot 1 + 4 \cdot 2 & 3 \cdot 1 + 1 \cdot 0 + 4 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 6 & -3 \\ 15 & 19 \end{bmatrix} \end{aligned}$$

Multiplying the same matrices, but in opposite order gives a 3×3-matrix:

$$\begin{aligned} BA &= \begin{bmatrix} 2{\cdot}5+1{\cdot}3 & 2{\cdot}0+1{\cdot}1 & 2{\cdot}(-2)+1{\cdot}4 \\ 1{\cdot}5+0{\cdot}3 & 1{\cdot}0+0{\cdot}1 & 1{\cdot}(-2)+0{\cdot}4 \\ 2{\cdot}5+4{\cdot}3 & 2{\cdot}0+4{\cdot}1 & 2{\cdot}(-2)+4{\cdot}4 \end{bmatrix} \\ &= \begin{bmatrix} 13 & 1 & 0 \\ 5 & 0 & -2 \\ 22 & 4 & 12 \end{bmatrix} \end{aligned}$$

The product of B and C is the 3×2-matrix

$$BC = \begin{bmatrix} 5 & 8 \\ 1 & 2 \\ 14 & 20 \end{bmatrix}$$

However the product CB is not defined, since C is of size 2×2 and B is of size 3×2.

We could have calculated CA, but AC is also not defined.

⚠ **Remark 5.3.3**

Another important remark is that cancellation of matrices are in general not possible. That is, if $AB = AC$, then we do **not** necessarily have $B = C$. For example

$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

However,

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \neq \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

Matrix transpose

Before moving on to the more interesting concept of inverse matrices, let us as an interlude mention a matrix operation which do not have an equal in number manipulation.

🛈 **Definition 5.3.2**

Let

$$\begin{bmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{bmatrix}$$

be a $m \times n$-matrix. The **transpose** of A is the $n \times m$-matrix

$$\begin{bmatrix} a_{1,1} & a_{2,1} & \cdots & a_{m,1} \\ a_{1,2} & a_{2,2} & \cdots & a_{m,2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1,n} & a_{2,n} & \cdots & a_{m,n} \end{bmatrix}$$

where the rows are the columns of A and the columns are the rows in A.

A square matrix A is **symmetric** if and only if $A = A^T$.

Example 5.3.8

If we let A be the matrix

$$A = \begin{bmatrix} 5 & 0 & -2 \\ 3 & 1 & 4 \end{bmatrix},$$

then the transpose of A is:

$$A^T = \begin{bmatrix} 5 & 3 \\ 0 & 1 \\ -2 & 4 \end{bmatrix}.$$

Theorem 5.3.4

For every matrix A we have:

$$(A^T)^T = A.$$

If A and B are matrices such that the product AB is defined, then:

$$(AB)^T = B^T \cdot A^T.$$

5.4 Invertability

For every number a not equal to zero, there exist an inverse number $a^{-1} = \frac{1}{a}$ such that $a \cdot a^{-1} = 1 = a^{-1}a$. It is natural to wonder whether this property extends to matrices. Let us limit the discussion to only square matrices, i.e. matrices of size $n \times n$ for some n. The question is then: Given a square $n \times n$-matrix A, can we find a matrix B such that the following equalities hold?

$$AB = I_n = BA.$$

Let us look at an example of such matrices.

Example 5.4.1

Let A and B be the following 2×2-matrices:

$$A = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \quad \text{og} \quad B = \begin{bmatrix} -1/2 & 1 \\ -3/2 & 2 \end{bmatrix}$$

Then we have

$$A \cdot B = \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} \begin{bmatrix} -1/2 & 1 \\ -3/2 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$B \cdot A = \begin{bmatrix} -1/2 & 1 \\ -3/2 & 2 \end{bmatrix} \begin{bmatrix} 4 & -2 \\ 3 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2.$$

These matrices therefore satisfies

$$A \cdot B = I_2 = B \cdot A.$$

Definition 5.4.1

Let A be a matrix of size $n \times n$. An **inverse** of A is a $n \times n$-matrix B such that

$$AB = I_n = BA.$$

A matrix is **invertible** if it has an inverse.

A square matrix do not necessarily have an inverse. The following three matrices are examples of such.

Example 5.4.2

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad \text{and} \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

are not invertible matrices.

Theorem 5.4.1

If a matrix has an inverse, then the inverse matrix is unique. We denote this matrix by A^{-1}.

One reason to be interested in the invertability of matrices comes from the solvability of linear equations. Remember that if we have a system of n linear equations in n variables, then we can construct the corresponding matrix equation $A\mathbf{x} = \mathbf{b}$ where A is the coefficient matrix.

Theorem 5.4.2

Let A be a $n \times n$-matrix, and $\mathbf{b} \in \mathbb{R}^n$ a vector. If A is invertible then $A\mathbf{x} = \mathbf{b}$ has a solution. The solution is unique and given by $A^{-1}\mathbf{b}$

Moreover, if $A\mathbf{x} = \mathbf{c}$ has a solution for every $\mathbf{c} \in \mathbb{R}^n$, then A is invertible.

Simultaneous solutions of multiple systems

Assume that we want to solve multiple matrix equations

$$\begin{aligned} A\mathbf{x}_1 &= \mathbf{b}_1 \\ A\mathbf{x}_2 &= \mathbf{b}_2 \\ &\vdots \\ A\mathbf{x}_t &= \mathbf{b}_t \end{aligned}$$

with the same coefficient matriz A, but different vectors on the right $\mathbf{b}_1$, $\mathbf{b}_2$, ..., $\mathbf{b}_t$. Then we could reduce the augmented matrix of each system individually:

$$\begin{gathered} \left[A \mid \mathbf{b}_1\right] \\ \left[A \mid \mathbf{b}_2\right] \\ \vdots \\ \left[A \mid \mathbf{b}_t\right] \end{gathered}$$

Or we could observe that since the left hand side is equal for each of them, we are doing the same thing multiple times except for what is on the right side. We can save ourselves a lot of trouble if we combine all the augmented matrices into one:

$$\left[\begin{array}{c|cccc} A & \mathbf{b}_1 & \mathbf{b}_2 & \cdots & \mathbf{b}_t \end{array}\right],$$

and reduce that instead.

Example 5.4.3

Let A be the following matrix:

$$A = \begin{bmatrix} 2 & -2 \\ 1 & 3 \end{bmatrix}$$

We would like to solve the following systems:

$$A\mathbf{x}_1 = \begin{bmatrix} -2 \\ 7 \end{bmatrix} \qquad A\mathbf{x}_2 = \begin{bmatrix} 10 \\ 1 \end{bmatrix} \qquad A\mathbf{x}_3 = \begin{bmatrix} -4 \\ 0 \end{bmatrix}$$

We combine them all into a big augmented matrix and reduce

$$\begin{aligned} \left[\begin{array}{cc|ccc} 2 & -2 & -2 & 10 & -4 \\ 1 & 3 & 7 & 1 & 0 \end{array}\right] &\sim \left[\begin{array}{cc|ccc} 1 & 3 & 7 & 1 & 0 \\ 2 & -2 & -2 & 10 & -4 \end{array}\right] \\ &\sim \left[\begin{array}{cc|ccc} 1 & 3 & 7 & 1 & 0 \\ 0 & -8 & -16 & 8 & -4 \end{array}\right] \\ &\sim \left[\begin{array}{cc|ccc} 1 & 3 & 7 & 1 & 0 \\ 0 & 1 & 2 & -1 & 1/2 \end{array}\right] \\ &\sim \left[\begin{array}{cc|ccc} 1 & 0 & 1 & 4 & -3/2 \\ 0 & 1 & 2 & -1 & 1/2 \end{array}\right] \end{aligned}$$

The last matrix is in reduced row echelon form, and we read the solutions of the systems from the right hand side of this matrix.

$$\mathbf{x}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{x}_2 = \begin{bmatrix} 4 \\ -1 \end{bmatrix} \qquad \mathbf{x}_3 = \begin{bmatrix} -3/2 \\ 1/2 \end{bmatrix}$$

Finding the inverse matrix

Now, let us find a formula for the inverse matrices. Assume that we have a $n \times n$-matrix A. We would ascertain whether it is invertible and if so find the inverse matrix A^{-1}.

We consider the equation
$$AX = I_n,$$
where X is an unknown $n \times n$-matrix. Let

$$\mathbf{x}_1, \ \mathbf{x}_2, \ \ldots, \ \mathbf{x}_n$$

be the columns of X, that is

$$X = \begin{bmatrix} \mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n \end{bmatrix},$$

then we can write the product AX as

$$AX = \begin{bmatrix} A\mathbf{x}_1 & A\mathbf{x}_2 & \cdots & A\mathbf{x}_n \end{bmatrix}$$

Let us name the columns of I_n as well:

$$I_n = \begin{bmatrix} \mathbf{e}_1 & \mathbf{e}_2 & \ldots & \mathbf{e}_n \end{bmatrix}$$

That means that $\mathbf{e}_i$ is the vector in $\mathbb{R}^n$ with 1 in the ith row, and 0's everywhere else.

The equation $AX = I_n$ is now equal to the equations:

$$\begin{aligned} A\mathbf{x}_1 &= \mathbf{e}_1 \\ A\mathbf{x}_2 &= \mathbf{e}_2 \\ &\vdots \\ A\mathbf{x}_n &= \mathbf{e}_n \end{aligned}$$

By the technique described above we can therefore reduce the augmented matrix

$$\left[\begin{array}{c|cccc} A & \mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n \end{array}\right] = \left[\begin{array}{c|c} A & I_n \end{array}\right]$$

to solve for X.

Example 5.4.4

We want to invert the following matrix if possible:

$$A = \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 3 \\ 2 & -2 & -1 \end{bmatrix}$$

Following the ideas above, we would like to find a matrix X such that $AX = I_3$

by solving the three systems:

$$A\mathbf{x}_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} \quad A\mathbf{x}_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} \quad A\mathbf{x}_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

We make the combined augmented matrix for the systems and reduce:

$$\left[\begin{array}{ccc|ccc}1 & -1 & -3 & 1 & 0 & 0\\0 & 1 & 3 & 0 & 1 & 0\\2 & -2 & -1 & 0 & 0 & 1\end{array}\right] \sim \left[\begin{array}{ccc|ccc}1 & -1 & -3 & 1 & 0 & 0\\0 & 1 & 3 & 0 & 1 & 0\\0 & 0 & 5 & -2 & 0 & 1\end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc}1 & -1 & -3 & 1 & 0 & 0\\0 & 1 & 3 & 0 & 1 & 0\\0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 1 & 0\\0 & 1 & 3 & 0 & 1 & 0\\0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}\right]$$
$$\sim \left[\begin{array}{ccc|ccc}1 & 0 & 0 & 1 & 1 & 0\\0 & 1 & 0 & \frac{6}{5} & 1 & -\frac{3}{5}\\0 & 0 & 1 & -\frac{2}{5} & 0 & \frac{1}{5}\end{array}\right]$$

We obtain from this the following solutions:

$$\mathbf{x}_1 = \begin{bmatrix}1\\6/5\\-2/5\end{bmatrix} \quad \mathbf{x}_2 = \begin{bmatrix}1\\1\\0\end{bmatrix} \quad \mathbf{x}_3 = \begin{bmatrix}0\\-3/5\\1/5\end{bmatrix}$$

Now, X is given by

$$X = \begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\6/5 & 1 & -3/5\\-2/5 & 0 & 1/5\end{bmatrix}$$

We have found X by solving $AX = I_3$, and if you try taking the product XA, you will find that this also equals I_3. We conclude that X is the inverse of A and A is invertible, that is $X = A^{-1}$

In the example we solved the equation $AX = I_3$, and it turned out that X also solved $XA = I_3$, so we concluded that $A^{-1} = X$. This actually holds in general.

Theorem 5.4.3

Let A be a $n \times n$-matrix.

(a) A is invertible if and only if the reduced row echelon form of A equals I_n.

(b) If A is invertible, we can find the inverse by reducing the matrix

$$\left[A \mid I_n\right]$$

to reduced row echelon form and read out the right hand side. In other word: The reduced row echelon form is on the form:

$$\left[I_n \mid A^{-1}\right]$$

Formula for inverting a 2×2-matrix

For a 2×2-matrix, there is an explicit formula for the inverse.

Theorem 5.4.4

Let

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

be invertible. Then

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Proof.

$$AA^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

The product $A^{-1}A = I_2$ is calculated in the same way. $\square$

Problem Set - Linear Equations and Matrices

1. Write the set of equations on matrix form. Find the augmented matrix and solve by Gaussian elimination.

a)

$$\begin{cases} x_1 - 2x_2 - 3x_3 = 0 \\ \quad 2x_2 + x_3 = -8 \\ -x_1 + x_2 + 2x_3 = 3 \end{cases}$$

b)

$$\begin{cases} 2x_1 + 3x_2 + 4x_3 = 13 \\ 3x_1 + 3x_2 + 6x_3 = 15 \end{cases}$$

c)

$$\begin{cases} x - 4y + 28z = -2 \\ -x + y - 7z = -31 \\ x + 2y - 14z = 64 \end{cases}$$

2. Which of these matrices are in row echelon form? Which of them are in reduced row echelon form?

a)

$$\begin{bmatrix} 1 & 5 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix}$$

c)

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix}$$

3.

a) Solve the set of equations

$$\begin{cases} 2x - y + z = 0 \\ 3x + y - 6z = 0 \\ 4x - 2y + 2z = 0 \end{cases}$$

and

$$\begin{cases} 2x - y + z = 1 \\ 3x + y - 6z = 4 \\ 4x - 2y + 2z = 2 \end{cases}$$

Explain the relation between the solutions.

b) Find values a, b, c such that

$$\begin{cases} 2x - y + z = a \\ 3x + y - 6z = b \\ 4x - 2y + 2z = c \end{cases}$$

don't have any solution.

4. Assume that we have given a set of equation consisting of m equations and n unknowns. Which of the nine cases in the following table is possible?

	$m < n$	$m = n$	$m > n$
0 solutions			
1 solution			
∞ solutions			

5. Determine whether the system of equations given by

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 5 \\ 1 & -2 & -1 & 1 \\ 0 & -4 & -1 & -1 \end{array}\right]$$

has a solution.

6. Consider a solved Sudoko puzzle as a 9×9-matrix A. Calculate the product

$$A \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$$

7. Let A and B be matrices, and $\mathbf{v}$ a vector:

$$A = \begin{bmatrix} 0 & 1 & 5 \\ 2 & 3 & -1 \\ -8 & 0 & 2 \end{bmatrix} \quad B = \begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \quad \mathbf{v} = \begin{bmatrix} 7 \\ 2 \\ -4 \end{bmatrix}$$

Calculate (or explain why the expression do not make sense):

a) AB d) B^2 g) $BA\mathbf{v}$

b) BA e) $A+B$ h) B^T

c) A^2 f) $(A+I_3)\mathbf{v}$ i) $\mathbf{v}^T\mathbf{v}$

8. Find a 2×2-matrix A such that

$$A \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 5 \end{bmatrix} \quad \text{and} \quad A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$$

9. Find, if possible, the inverse matrices of

a)

$$\begin{bmatrix} 1 & 7 \\ -1 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

10.

a) For which values of $a \in \mathbb{R}$ is the matrix

$$A = \begin{bmatrix} 1 & 0 & a \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix}$$

invertible?

b) Verify that

$$A^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 & -1 \\ -1 & 1 & 1 \\ 1 & -1 & 1 \end{bmatrix}$$

when $a = 1$.

Solutions - Linear Equations and Matrices

1. a The system of equations can be written on the form $A\mathbf{x} = \mathbf{b}$, with

$$A = \begin{bmatrix} 1 & -2 & -3 \\ 0 & 2 & 1 \\ -1 & 1 & 2 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -8 \\ 3 \end{bmatrix}$$

Gauss elimination yields

$$\left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ -1 & 1 & 2 & 3 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 2 & 1 & -8 \\ 0 & -1 & -1 & 3 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 2 & 1 & -8 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -2 & -3 & 0 \\ 0 & 1 & 1 & -3 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

and we obtain $x + 3$, $x_2 = -5$ and $x_1 = -4$ by backward substitution. The solution

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -4 \\ -5 \\ 2 \end{bmatrix}$$

is unique.

1. b We have

$$A = \begin{bmatrix} 2 & 3 & 4 \\ 3 & 3 & 6 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 13 \\ 15 \end{bmatrix}$$

By Gaussian elimination we get

$$\left[\begin{array}{ccc|c} 2 & 3 & 4 & 13 \\ 3 & 3 & 6 & 15 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & 3 & 4 & 13 \\ 1 & 0 & 2 & 2 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 3 & 0 & 9 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 3 \end{array}\right]$$

we can choose x_3 as we want, let's say $x_3 = t$, and then we get $x_1 = 2 - 2t$ and $x_2 = 3$. The number of solutions are infinite, and they are on the form

$$\mathbf{x} = \begin{bmatrix} 2 - 2t \\ 3 \\ t \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 0 \end{bmatrix} + t \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$$

for $t \in \mathbb{R}$.

1. c The system written as an augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & -4 & 28 & -2 \\ -1 & 1 & -7 & -31 \\ 1 & 2 & -14 & 64 \end{array}\right]$$

By Gaussian elimination we get

$$\left[\begin{array}{ccc|c} 1 & -4 & 28 & -2 \\ -1 & 1 & -7 & -31 \\ 1 & 2 & -14 & 64 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 42 \\ 0 & 1 & -7 & 11 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

From row 1 we read directly that $x = 42$. Row 3 means

$$0x + 0y + 0z = 0$$

or $0 = 0$. This do not tell us anything about the values of x, y and z. We have a free variable $z = t$, where t is a real number (we could also have chosen y to be the free variable). With $z = t$ we see in row 2 that $y = 11 + 7t$. The number of solutions are infinite, and they are on the form

$$\mathbf{x} = \begin{bmatrix} 42 \\ 11 + 7t \\ t \end{bmatrix} = \begin{bmatrix} 42 \\ 11 \\ 0 \end{bmatrix} + t \begin{bmatrix} 0 \\ 7 \\ 1 \end{bmatrix}$$

for $t \in \mathbb{R}$.

2. Matrix (a) and (c) are in row echelon form; (a) is in row reduced echelon form.

3. a We put the homogeneous system into an augmented matrix and eliminate:

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 3 & 1 & -6 & 0 \\ 4 & -2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 0 & \frac{5}{2} & -\frac{15}{2} & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 0 \\ 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The third variable is free, we put it as $x_3 = t$. Then we see in the second row that $x_2 = 3t$, and in the first row that $x_1 = t$. We have the solution

$$x = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} t, \quad \forall t \in \mathbb{R}.$$

In the next system, the inhomogeneous one, we get a nearly identical augmented matrix, except in the fourth column:

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 3 & 1 & -6 & 4 \\ 4 & -2 & 2 & 2 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & \frac{5}{2} & -\frac{15}{2} & \frac{5}{2} \\ 0 & 0 & 0 & 0 \end{array}\right]$$
$$\sim \left[\begin{array}{ccc|c} 1 & -\frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & -1 & 1 \\ 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Again, we obtain a free variable $x_3 = t$. The second row then tells us that $x_2 = 1 + 3t$, and the first row that $x_1 = 1 + t$. In vector notation, the solution

$$x = \begin{bmatrix} 1+t \\ 1+3t \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix} t, \quad \forall t \in \mathbb{R}.$$

3. b We must find a vector $\mathbf{b} = [b_1, b_2, b_3]^T$ such that the system $A\vec{x} = \mathbf{b}$ can't be solved, where

$$A = \begin{bmatrix} 2 & -1 & 1 \\ 3 & 1 & -6 \\ 4 & -2 & 2 \end{bmatrix}$$

This happens if and only if we have an impossible equation in the reduced system. There are a lot of possibilities, but we notice that in the third row of our matrix, which represent $4x - 2y + 2z$, is two times the first row. So let us try a vector $\mathbf{b}$ where b_3 is not equal to $2b_1$? For example $\mathbf{b} = [0, 0, 1]^T$,

$$\left[\begin{array}{ccc|c} 2 & -1 & 1 & 0 \\ 3 & 1 & -6 & 0 \\ 4 & -2 & 2 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 2 & -1 & 1 & 1 \\ 0 & \frac{5}{2} & -\frac{15}{2} & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Here we have a contradiction in the third row, for no values of x, y, z can solve $0x + 0y + 0z = 1$. The vector $\mathbf{b} = [0, 0, 1]^T$ is therefore one of many possible answers to the problem.

4.

Let 1 denote possible and 0 impossible:

	$m < n$	$m = n$	$m > n$
0 solutions	1	1	1
1 solution	0	1	1
∞ solutions	1	1	1

Explanation: In each case we either find an example, or explain why no examples exists.

No solutions: Regardless the amount of equations and unknowns, we can always construct an example with an equation that says $0 = 1$.
Infinitely many solutions: We need to verify that we can always construct an example with a free variable. In the case $m < n$ we can take the 1×2 system $x + y = 0$, where y is free. For $m = n$ we can add the equation $0 = 0$ so we have two equations with two unknowns, and y still free. For $m > n$ we can add the equation $0 = 0$ again.
Exactly one solution: For $m = n$ we can take the system $x = 1$, which has a unique solution. If you don't accept this as a proper system you can add on $y = 1$. To obtain an example for $m > n$ we add on the equation $0 = 0$. Now, we have to think, "What happens if we have more unknowns than equations?". We can construct the augmented matrix of the system, this is wider than it is high, since we have more unknowns than equation. Thus we can't get a pivot element in each column. Each column without a pivot element gives a free variable. Hence the system has either no solutions or infinitely many.

5.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 5 \\ 1 & -2 & -1 & 1 \\ 0 & -4 & -1 & -1 \end{array}\right] \xrightarrow{III=III+II} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 1 & 2 & 3 & 5 \\ 2 & 0 & 2 & 6 \\ 0 & -4 & -1 & -1 \end{array}\right]$$

If we add row two to row three, we see that we have $x + y = 0$ and $2x + 2y = 6$, which can't both be true. Thus the system has no solution.

6. In a Sudoku puzzle every row has to contain exactly one copy of each number between 1 and 9, so when you multiply with $\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix}^T$, you will in each term get a sum

$$1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

in each entry, thus

$$A \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 45 \\ 45 \\ 45 \\ 45 \\ 45 \\ 45 \\ 45 \\ 45 \\ 45 \end{bmatrix}$$

Our solved Sudoku puzzle could for example be

9	4	8	5	7	6	3	2	1
7	5	1	9	3	2	4	8	6
3	6	2	4	8	1	5	7	9
8	1	7	6	9	5	2	3	4
5	2	9	7	2	4	1	6	8
4	2	6	3	1	8	9	5	7
6	9	5	2	4	7	8	1	3
2	8	4	1	6	3	7	9	5
1	7	3	8	5	9	6	4	2

7.

a) Mismatching sizes.

b) $\begin{bmatrix} -36 & 7 & 13 \\ -24 & 0 & 6 \end{bmatrix}$

c) $\begin{bmatrix} -38 & 3 & 9 \\ 14 & 11 & 5 \\ -16 & -8 & -36 \end{bmatrix}$

d) Mismatching sizes.

e) Mismatching sizes.

f) $\begin{bmatrix} -11 \\ 26 \\ -68 \end{bmatrix}$

g) $\begin{bmatrix} -290 \\ -192 \end{bmatrix}$

h) $\begin{bmatrix} 1 & 0 \\ 2 & 0 \\ 5 & 3 \end{bmatrix}$

i) 69

8. $A \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is the first column of A, while $A \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is the sum of the columns in A, i.e.

$$A = \begin{bmatrix} 3 & x \\ 5 & y \end{bmatrix}$$

where $3 + x = -1$ og $5 + y = 0$. Thus

$$A = \begin{bmatrix} 3 & -4 \\ 5 & -5 \end{bmatrix}$$

9. a We put the identity matrix on the side of the matrix, and reduce it to reduced row echelon form:

$$\begin{bmatrix} 1 & 7 & 1 & 0 \\ -1 & 1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 7 & 1 & 0 \\ 0 & 8 & 1 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & \frac{1}{8} & \frac{-7}{8} \\ 0 & 1 & \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

so the inverse is

$$\begin{bmatrix} \frac{1}{8} & \frac{-7}{8} \\ \frac{1}{8} & \frac{1}{8} \end{bmatrix}$$

9. b The method from a) gives the inverse

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}.$$

10. a The matrix A is invertible whenever it has maximum rank (which is 3). By Gaussian elimination we find

$$\begin{bmatrix} 1 & 0 & a \\ a & 1 & 0 \\ 0 & a & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & a^2 \\ 0 & a & 1 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & a \\ 0 & 1 & -a^2 \\ 0 & 0 & 1+a^3 \end{bmatrix}$$

From the echelon form we see that A has maximum rank if and only if $1 + a^3 \neq 0$, thus whenever $a \neq -1$. Thus the matrix is invertible if and only if $a \neq -1$.

10. b Multiply the two matrices and check that you get the identity matrix.

Chapter 6

Linear independence and subspaces

6.1 Linear independence

Linear independence in $\mathbb{R}^3$

If we have to vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ in $\mathbb{R}^3$, both non-zero, there are two possibilities

I. They are parallel, i.e. $\mathbf{v_1} = a\mathbf{v_2}$ for some scalar $a \neq 0$. then

$$\operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}\} = \operatorname{span}\{\mathbf{v_1}\} = \operatorname{span}\{\mathbf{v_2}\}.$$

and we say that the two vectors are **linearly dependent**.

II. They are *not* parallel. Then

$$\operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}\}$$

is a plane in $\mathbb{R}^3$ containing the origin. We say that the two vectors are **linearly independent.**

In case II, we introduce a third non-zero vector $\mathbf{v_3}$. Once again we have two possibilities:

II(a) $\mathbf{v_3}$ lies in the plane spanned by $\mathbf{v_1}$ and $\mathbf{v_2}$, i.e. $\operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}\}$. Equivalently $\mathbf{v_3} = a\mathbf{v_1} + b\mathbf{v_2}$. All vectors lie in the same plane, and we say that the vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly dependent.

II(b) $\mathbf{v_3}$ do *not* lie in the plane spanned by $\mathbf{v_1}$, and $\mathbf{v_2}$. Then $\operatorname{span}\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}\}$ is the whole of $\mathbb{R}^3$, and the three vectors $\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}$ are linearly independent.

Linear independence in $\mathbb{R}^n$

Definition 6.1.1

The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ are **linearly independent** if the equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots + x_n\mathbf{v}_n = \mathbf{0}$$

only have the trivial solution $x_1 = x_2 = \cdots = x_n = 0$. Otherwise they are **linearly dependent**.

A set of vectors are linearly independent only if the only way of combining them into the zero vector is to choose all scales to be zero.

Example 6.1.1

Consider

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

in $\mathbb{R}^3$. The equation

$$\mathbf{v}_1 x_1 + \mathbf{v}_2 x_2 = \mathbf{0}$$

that is

$$\begin{bmatrix} 3 \\ -2 \\ 0 \end{bmatrix} x_1 + \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} x_2 = \begin{bmatrix} 3x_1 \\ -2x_1 \\ 4x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

gives $3x_1 = 0$ og $4x_2 = 0$, so $x_1 = x_2 = 0$. The equation only has the trivial solution, so $\mathbf{v}_1$ and $\mathbf{v}_2$ are linear independent.

Example 6.1.2

Let $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ be the following three vectors in $\mathbb{R}^3$:

$$\mathbf{u} = \begin{bmatrix} 4 \\ 4 \\ 9 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

These vectors satisfy the following equality:

$$\mathbf{u} + 5 \cdot \mathbf{v} - 9 \cdot \mathbf{w} = \mathbf{0}$$

and is therefore linear dependent.

Checking for independence

To check if a collection of vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ are linearly independent, we solve the equation

$$\mathbf{v}_1 \cdot x_1 + \mathbf{v}_2 \cdot x_2 + \cdots + \mathbf{v}_n \cdot x_n = \mathbf{0}$$

The left hand side can be recognized as a matrix with columns $\mathbf{v}_i$ multiplied with a vector with elements x_i. The equation is therefore equal to

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

This can be solved as usual by reducing the augmented matrix

$$\left[\begin{array}{cccc|c} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n & \mathbf{0} \end{array}\right]$$

Example 6.1.3

Are these vectors linear independent?

$$\mathbf{u} = \begin{bmatrix} 3 \\ 9 \\ 3 \\ 3 \end{bmatrix}, \qquad \mathbf{v} = \begin{bmatrix} 2 \\ 7 \\ 2 \\ 4 \end{bmatrix} \qquad \text{and} \qquad \mathbf{w} = \begin{bmatrix} 8 \\ 31 \\ 12 \\ 22 \end{bmatrix}$$

The equation is

$$\mathbf{u} \cdot x + \mathbf{v} \cdot y + \mathbf{w} \cdot z = \mathbf{0},$$

Reducing the augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & 2 & 8 & 0 \\ 9 & 7 & 31 & 0 \\ 3 & 2 & 12 & 0 \\ 3 & 4 & 22 & 0 \end{array}\right] \sim \left[\begin{array}{ccc|c} 3 & 2 & 8 & 0 \\ 0 & 1 & 7 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that we will obtain a pivot element in each of the columns, and thus no free variables. Then we only have one solution: $x = y = z = 0$. We conclude that $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$ are linearly independent.

The essential information in the reduction to establish linear (in)dependence is the existence of free variables. Do note that we do not need to include the right hand side in the reduction. There are only zeros in the right hand side from the beginning, so they will stay zero regardless of the row operation done.

Theorem 6.1.1

Let A be a matrix. The following statements are equivalent:

1. The columns of A are linearly independent.
2. The equation $A\mathbf{x} = \mathbf{0}$ is only solved by the trivial solution $\mathbf{x} = \mathbf{0}$.
3. There are no free variables in the solution of $A\mathbf{x} = \mathbf{0}$.
4. When we reduce A, we get a pivot element in every column.

Remark 6.1.1

Theorem 6.1.1 gives us a good method of checking for linear indepence. If we have vectors

$$\mathbf{v}_1,\ \mathbf{v}_2,\ \ldots,\ \mathbf{v}_n,$$

We check for indepence by:

1. Constructing the matrix $A = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n\end{bmatrix}$ having the vectors as columns.
2. Reduce A to reduced row echelon form.
3. If every column has a pivot element we conclude that the vectors are linearly independent. Otherwise they are linearly dependent

Example 6.1.4

Are these vectors linearly dependent?

$$\mathbf{v}_1 = \begin{bmatrix}5\\10\\5\\0\end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix}3\\7\\5\\4\end{bmatrix} \qquad \text{og} \qquad \mathbf{v}_3 = \begin{bmatrix}2\\6\\6\\8\end{bmatrix}$$

We reduce the matrix $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3\end{bmatrix}$:

$$\begin{bmatrix} 5 & 3 & 2 \\ 10 & 7 & 6 \\ 5 & 5 & 6 \\ 0 & 4 & 8 \end{bmatrix} \sim \begin{bmatrix} 5 & 3 & 2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

The last column do not contain a pivot element. Hence $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly dependent.

Example 6.1.5

Are these vectors linearly dependent?

$$\mathbf{v}_1 = \begin{bmatrix} 8 \\ 7 \\ 4 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 14 \\ -2 \\ 5 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} 7 \\ 5 \\ 11 \end{bmatrix}$$

We could reduce the following matrix:

$$\begin{bmatrix} 8 & 14 & 3 & 7 \\ 7 & -2 & 1 & 5 \\ 4 & 5 & 0 & 11 \end{bmatrix}$$

However, we do not need to in this case. We observe that we can't get more than three pivot elements. Thus there can't be a pivot element in all the four columns. The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$ and $\mathbf{v}_4$ are linearly dependent.

Theorem 6.1.2

Given n vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ in $\mathbb{R}^m$. If

1. one of the vectors is a linear combination of the other, or
2. $n > m$,

then the vectors are linearly dependent.

Theorem 6.1.3

If we have n vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_n$ in $\mathbb{R}^n$, then they are linearly independent if

> and only if they span the whole of $\mathbb{R}^n$, that is if and only if
>
> $$\operatorname{span}\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n\} = \mathbb{R}^n.$$

Proof. Let

$$A = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2 & \ldots & \mathbf{v}_n\end{bmatrix}$$

be the $n \times n$-matrix with our vectors as columns. We know that the vectors are linearly independent if and only if we obtain a pivot element in every column when we reduce A. Now, since A is a square matrix this is the same as having pivot elements in every row. That is equivalent with

$$A\mathbf{x} = \mathbf{b}$$

having solutions for all vectors $\mathbf{b}$ in $\mathbb{R}^n$, which says that the columns of A span $\mathbb{R}^n$. □

6.2 Subspaces

We have until now been talking about the **Euclidean vector spaces** $\mathbb{R}^n$, but there also exist a whole lot more vector spaces. For example, if we instead of looking at the product of real number lines

$$\mathbb{R}^n = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n \text{ copies}}$$

look at the product of complex number lines

$$\mathbb{C}^n = \underbrace{\mathbb{C} \times \mathbb{C} \times \cdots \times \mathbb{C}}_{n \text{ copies}}$$

we obtain a new vector space $\mathbb{C}^n$ consisting of vectors with complex entries

$$\mathbf{z} = \begin{bmatrix} z_1 \\ z_2 \\ \vdots \\ z_n \end{bmatrix} = \begin{bmatrix} a_1 + ib_1 \\ a_2 + ib_2 \\ \vdots \\ a_n + ib_n \end{bmatrix}.$$

Here the scalar multiplication is given component wise with complex numbers and addition of vectors is still defined component wise. More abstract examples of vector spaces and the defining properties of them can be found in section **??**.

For now we will be interested in a more nicely behaved kind of vector space, namely **subspaces** of Euclidean vector spaces.

Definition 6.2.1

A **subspace** of $\mathbb{R}^n$ is a subset of vectors $U \subseteq \mathbb{R}^n$ such that

1. The zero vector is contained in U,
$$\mathbf{0} \in U.$$
2. U is closed under scalar multiplication, i.e. if $\mathbf{u}$ is a vector in U and a is any scalar, then $a\mathbf{u}$ also lies in U,
$$\mathbf{u} \in U \text{ and } a \in \mathbb{R} \implies a\mathbf{u} \in U.$$
3. U is closed under vector addition, i.e. if $\mathbf{u}$ and $\mathbf{v}$ are vectors in U, then their sum $\mathbf{u}+\mathbf{v}$ is also in U,
$$\mathbf{u}, \mathbf{v} \in U \implies \mathbf{u}+\mathbf{v} \in U.$$

We have already seen examples of such subspaces. The linear span of a set of vectors is a subspace.

Example 6.2.1

Let $\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t$ be t vectors in $\mathbb{R}^n$, then

$$\operatorname{span}(\mathbf{x}_1, \mathbf{x}_2, \ldots, \mathbf{x}_t) = \{a_1\mathbf{x}_1 + a_2\mathbf{x}_2 + \cdots + a_t\mathbf{x}_t \mid a_1, \ldots, a_t \in \mathbb{R}\}$$

is a subspace.

In the last section we studied linear dependency. As a continuation of that we will now define the dimension of a subspace.

Definition 6.2.2

The **dimension** of a subspace U, denoted $\dim(U)$, is the maximum number of linearly independent vectors in U.
That is to say, if there exists n vectors $v_1, \ldots v_n \in U$ that are linearly independent, but $n+1$ vectors $\mathbf{v}_1, \ldots \mathbf{v}_{n+1} \in U$ are always linearly dependent, then the dimension $\dim U = n$. The dimension of $\{0\}$ is 0.

A **basis** of U is a maximal (dimension n) set of linearly independent vectors. If a set $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is a basis of U, then any vector in U can be represented as a unique linear combination of this basis. i.e. for any $\mathbf{u} \in U$ there exists unique scalars $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ such that

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \cdots + \alpha_n \mathbf{v}_n.$$

Example 6.2.2

The vector space $\mathbb{R}^2$ has dimension 2 and the two vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

gives a basis of $\mathbb{R}^2$.

The vector space $\mathbb{R}^3$ has dimension 3 and the three vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

gives a basis of $\mathbb{R}^3$.

The vector space $\mathbb{R}^n$ has dimension n and one basis is

$$\left(\begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \right)$$

The bases given above are called the standard bases of $\mathbb{R}^2$, $\mathbb{R}^3$ and $\mathbb{R}^n$.

Subspaces of a matrix

Definition 6.2.3

The **null space** of A is the space of all solutions of the homogeneous equation $A\mathbf{x} = \mathbf{0}$:

$$\mathrm{Null} A = \{\mathbf{x} \in \mathbb{R}^n \mid A\mathbf{x} = \mathbf{0}\}$$

Definition 6.2.4

The **column space** of a real $m \times n$-matrix

$$A = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \cdots & \mathbf{a}_n \end{bmatrix}$$

is the subspace of $\mathbb{R}^m$ spanned by the columns of A:

$$\text{Col}A = \text{span}\{\mathbf{a}_1, \mathbf{a}_2, \ldots, \mathbf{a}_n\}$$

The column space of A is the collection of all linear combination of the columns in A. A product $A\mathbf{v}$ of the matrix A and a vector $\mathbf{v}$ in $\mathbb{R}^n$ is exactly a linear combination of the columns in A, so we can descrive the column space as all vectors on the form $A\mathbf{v}$:

$$\text{Col}A = \{A\mathbf{v} \mid \mathbf{v} \in \mathbb{R}^n\}$$

Definition 6.2.5

Therow space of a real $m \times n$-matrix

$$A = \begin{bmatrix} \mathbf{r}_1^T \\ \mathbf{r}_2^T \\ \vdots \\ \mathbf{r}_m^T \end{bmatrix}$$

is the subspace of $\mathbb{R}^n$ spanned by the rows in A:

$$\text{Row}A = \text{span}\{\mathbf{r}_1, \mathbf{r}_2, \ldots, \mathbf{r}_m\}$$

The row space of A is all the linear combination of the rows in A (considering the rows as columns). This is equal to the column space of the transposed matrix:

$$\text{Row}A = \text{Col}A^T$$

Example 6.2.3

Let A be the following matrix:

$$A = \begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & 3 & 7 & 1 \\ 1 & 2 & 2 & 5 & 1 \\ 3 & 6 & 6 & 15 & 3 \end{bmatrix}$$

Let us try to descrive the null-space, column space and row space of A.

To find the null-space, we solve the equation

$$A\mathbf{x} = \mathbf{0}.$$

We do that by reducing A:

$$\begin{bmatrix} 1 & 2 & 1 & 2 & 0 \\ 2 & 4 & 3 & 7 & 1 \\ 1 & 2 & 2 & 5 & 1 \\ 3 & 6 & 6 & 15 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 & -1 & -1 \\ 0 & 0 & 1 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

We obtain three free variables, and the general solution is:

$$\mathbf{x} = r \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

and

$$\mathrm{Null} A = \mathrm{span}\left(\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \\ 1 \end{bmatrix} \right).$$

The vectors above is also linearly independent, so they form a basis for the null space

Directly from the definitions we have that the column and row space is given as the following sets:

$$\mathrm{Col} A = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 2 \\ 7 \\ 5 \\ 15 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 3 \end{bmatrix} \right)$$

$$\mathrm{Row} A = \mathrm{span}\left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \\ 7 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 2 \\ 5 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 6 \\ 6 \\ 15 \\ 3 \end{bmatrix} \right)$$

However, we might like to have basis and not only a spanning set. It turns out that we can find these bases by reducing A.

We start by considering the column space. In the reduced row echelon form of A, we have pivot elements in the first and third column. If we switch around the columns in A such that the first and third column is placed in front and do the corresponding switches in the reduced row form, we get:

$$\left[\begin{array}{cc|ccc} 1 & 1 & 2 & 2 & 0 \\ 2 & 3 & 4 & 7 & 1 \\ 1 & 2 & 2 & 5 & 1 \\ 3 & 6 & 6 & 15 & 3 \end{array}\right] \sim \left[\begin{array}{cc|ccc} 1 & 0 & 2 & -1 & -1 \\ 0 & 1 & 0 & 3 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Denote the columns of A by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$. The reduced row form now tells us that the columns without pivot elements, that is $\mathbf{a}_2$, $\mathbf{a}_4$ and $\mathbf{a}_5$, can be written as linear combinations of the pivot columns, $\mathbf{a}_1$ og $\mathbf{a}_3$:

$$\mathbf{a}_2 = 2 \cdot \mathbf{a}_1, \;\; \mathbf{a}_4 = (-1) \cdot \mathbf{a}_1 + 3\mathbf{a}_3, \;\; \mathbf{a}_5 = (-1) \cdot \mathbf{a}_1 + 1 \cdot \mathbf{a}_3.$$

Specifically, the columns without pivot elements is in the subspace spanned by the pivot columns,

$$\mathbf{a}_2, \mathbf{a}_4, \mathbf{a}_5 \in \operatorname{span}\{\mathbf{a}_1, \mathbf{a}_3\}.$$

The space spanned by all the columns in A, that is $\mathrm{Col}(A)$, is therefore equal to the space spanned by only $\mathbf{a}_1$ og $\mathbf{a}_3$. In addition, the reduced row form of A shows that $\mathbf{a}_1$ and $\mathbf{a}_3$ are linearly independent.

This shows that

$$(\mathbf{a}_1, \mathbf{a}_3) = \left(\begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 2 \\ 6 \end{bmatrix} \right)$$

is a basis of the column space $\mathrm{Col}A$.

It is somewhat easier to realize how to find a basis of the row space. We observe that if we perform an elementary row operation on a matrix, we still have the same row space. This follows from the fact that the rows in the resulting matrix from a row operation are linear combinations of the rows in the old matrix. Also, we can always find a row operation that gives us the old matrix from the new one.

Hence, to describe the row space of A it is enough to consider the row reduced form of A. Here we can easily observe that every non-zero row is necessarily linearly independent. We have that

$$\left(\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 3 \\ 1 \end{bmatrix} \right)$$

is a basis of the row space $\mathrm{Row}A$.

Remark 6.2.1

The example above shows us a method of constructing a basis for the column space of a matrix A:

1. Reduce A to reduced row echelon form.
2. Identify the pivot columns.
3. The pivot columns gives a basis for $\mathrm{Col}A$.

Warning: Rememeber that in the last step we need to use the columns in the original matrix A. Row operations do change the column space.

Theorem 6.2.1

Let A be an $m \times n$-matrix, and let E be the reduced row form of A. Then:

(a) The dimension of the null space, $\mathrm{Null}A$, is equal to the number of free variables we get when solving $A\mathbf{x} = \mathbf{0}$, which is equal to the amount of columns without pivot elements in E.

(b) The dimension of the column space, $\mathrm{Col}A$, is equal to the amount of columns containing pivot elements in E.

(c) The dimension of the row space, $\mathrm{Row}(A)$, equals the amount of rows not equal to zero in E.

Theorem 6.2.2

Let A be a matrix. The column and row space of A has equal dimension

$$\dim \mathrm{Col}A = \dim \mathrm{Row}A$$

This number, which is the dimension of both the column and row space is called the **rank** of A.

$$\text{rank}A = \dim \text{Col}A = \dim \text{Row}A$$

Since every column in the reduced row form either contains a pivot element or gives a free variable for the equation $A\mathbf{x} = \mathbf{0}$, we obtain the following result from theorem 6.2.1.

Theorem 6.2.3

Let A be a $m \times n$-matrix. Then

$$\dim \text{Null}A + \text{rank}A = n.$$

Let us end this section by combining these results with results on invertability

Theorem 6.2.4

Let A be a $n \times n$-matrix. The following statements are equivalent

1. A is invertible
2. The columns of A are linearly independent
3. The rows of A are linearly independent
4. $\text{rank}A = n$
5. $A\mathbf{x} = \mathbf{b}$ has a unique solution $(\mathbf{x} = A^{-1}\mathbf{b})$
6. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{c}$.
7. The reduced row echelon form of A is I_n
8. A has a pivot element in every row
9. $A\mathbf{x} = \mathbf{0}$ has only one solution, $\mathbf{x} = \mathbf{0}$
10. $\text{Col}(A) = \mathbb{R}^n$
11. $\text{Row}(A) = \mathbb{R}^n$

6.3 Determinant

One of the sought after properties of a square matrix $(n \times n)$ is invertability. If a matrix A is invertible, then we know that every equation $A\mathbf{x} = \mathbf{b}$ has a unique solution $x = A^{-1}\mathbf{b}$. We will now explore a number which determines invertability, namely **the determinant.** If we have $n \times n$-matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix},$$

then the determinant of A is denoted by either

$$\det A \qquad \text{or} \qquad \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.$$

Determinants of 2×2-matrices

The determinant of a 2×2-matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is defined as:

$$\det A = \begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

Example 6.3.1

$$\det \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} = 1 \cdot 4 - 2 \cdot 3 = 4 - 9 = -5$$

Determinants of 3×3-matrices

For a 3×3-matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix},$$

we define the determinant to be:

$$\det A = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$
$$= a_{11} \begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix} - a_{12} \begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix} + a_{13} \begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

We will present the general rule for arbitrary large matrices in a bit, but let us observe that the rules at the moment are:

- the sign switches as $+ - + - +-$ horizontally (addition first)
- the smaller "sub-determinants" are obtained by deleting the row and column which the coefficient of it lie in, i.e.

the "sub-determinant"

$$\begin{vmatrix} a_{22} & a_{23} \\ a_{32} & a_{33} \end{vmatrix}$$

is found by deleting the row and column containing a_{11}:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

likewise, we obtain

$$\begin{vmatrix} a_{21} & a_{23} \\ a_{31} & a_{33} \end{vmatrix}$$

through deleting the first row and second column:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

and lastly

$$\begin{vmatrix} a_{21} & a_{22} \\ a_{31} & a_{32} \end{vmatrix}$$

is obtained by deleting the first row and third column:

$$\begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

Example 6.3.2

$$\begin{aligned} \det \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix} &= 1 \cdot \begin{vmatrix} 5 & 6 \\ 8 & 9 \end{vmatrix} - 2 \cdot \begin{vmatrix} 4 & 6 \\ 7 & 9 \end{vmatrix} + 3 \cdot \begin{vmatrix} 4 & 5 \\ 7 & 8 \end{vmatrix} \\ &= 1(45-48) - 2(36-42) + 3(32-35) \\ &= -3 + 12 - 9 = 0 \end{aligned}$$

Determinants: general definition

We define the determinant of an arbitrary big quadratic matrix in the same pattern as the determinant of a 3×3-matrix.

Definition 6.3.1

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix}$$

be an $n \times n$-matrix. The **determinant** of A is defined as follows.

1. If $n = 1$, then $A = \begin{bmatrix} a_{11} \end{bmatrix}$, and we define $\det A = a_{11}$.

2. If $n > 1$, we introduce a few helping variables. For every i and j from 1 to n we set A_{ij} to be the $(n-1) \times (n-1)$-matrix we obtaing by remocing row i and column j from A, and we define

$$C_{ij} = (-1)^{i+j} \det A_{ij}$$

to be the determinant of this matrix with a sign dependent on i and j. The determinant of A is defined as:

$$\det A = \sum_{j=1}^{n} a_{1j} C_{1j}$$

The numbers C_{ij} in the definition are called **cofactors** of A.

One can now see that this definition reduces back to our already defined cases for 2×2-matrices and 3×3-matrices.

Example 6.3.3

Let A be the following 4×4-matrix:

$$A = \begin{bmatrix} 3 & 0 & 2 & 4 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 5 & 3 \end{bmatrix}$$

We calculate the determinant of A. From the definition we have:

$$\begin{aligned} \det A = 3 \begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 3 \end{vmatrix} - 0 \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} \\ + 2 \begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 7 & 0 & 3 \end{vmatrix} - 4 \begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 7 & 0 & 5 \end{vmatrix} \end{aligned}$$

We calculate each of the 3×3-determinants needed (Note that we do not calculate the second, since it will be multiplied by zero):

$$\begin{vmatrix} 2 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 3 \end{vmatrix} = 2 \begin{vmatrix} 1 & 0 \\ 5 & 3 \end{vmatrix} - 1 \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} + 0 \begin{vmatrix} 0 & 1 \\ 0 & 5 \end{vmatrix} = 6$$

$$\begin{vmatrix} 1 & 2 & 0 \\ 2 & 0 & 0 \\ 7 & 0 & 3 \end{vmatrix} = 1 \begin{vmatrix} 0 & 0 \\ 0 & 3 \end{vmatrix} - 2 \begin{vmatrix} 2 & 0 \\ 7 & 3 \end{vmatrix} + 0 \begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} = -12$$

$$\begin{vmatrix} 1 & 2 & 1 \\ 2 & 0 & 1 \\ 7 & 0 & 5 \end{vmatrix} = 1 \begin{vmatrix} 0 & 1 \\ 0 & 5 \end{vmatrix} - 2 \begin{vmatrix} 2 & 1 \\ 7 & 5 \end{vmatrix} + 1 \begin{vmatrix} 2 & 0 \\ 7 & 0 \end{vmatrix} = -6$$

Substituting these values into the expression for $\det A$ we have:

$$\det A = 3 \cdot 6 - 0 + 2 \cdot (-12) - 4 \cdot (-6) = 18$$

Cofactor-expansion

In the definition of the determinant we move through the first row in the matrix and look at the numbers

$$a_{11},\ a_{12},\ \ldots,\ a_{1n}.$$

Each number a_{1j} is multiplied with the associated cofactor C_{1j}, and at the end we add all these products

However, we can also move along any other row with the same procedure and get the same end product. Also, we can move along any column.

Theorem 6.3.1

Let A be a $n \times n$-matrix, where $n > 1$, and let A_{ij} and C_{ij} be given as in the definition of the determinant. Then we have

$$\det A = \sum_{j=1}^{n} a_{kj} C_{kj} = \sum_{i=1}^{n} a_{il} C_{il}$$

for all k and l such that $1 \leq k \leq n$ and $1 \leq l \leq n$.

The method of finding the determinant through cofactors as in the theorem is called **cofactorexpansion**. We say that we do the cofactor-expansion along row k or along column l.

Let us now find the same determinant as in example 6.3.3, but in a more clever way.

We have to make sure that the signs in the cofactors are correct. When we expand along the first row (as in the definition), or along the first column, we always start with a positive sign in the first part. However, when we expand along a different row or column, it might be that we start with a negative sign. A good rule of thumb is to use the following diagram to remember which sign each cofactor has:

$$\begin{bmatrix} + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ + & - & + & - & \cdots \\ - & + & - & + & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Example 6.3.4

Once again, let A be the following 4×4-matrix:

$$A = \begin{bmatrix} 3 & 0 & 2 & 4 \\ 1 & 2 & 1 & 0 \\ 2 & 0 & 1 & 0 \\ 7 & 0 & 5 & 3 \end{bmatrix}$$

We observe that the second column consists almost entirely of zeros, so it can be smart to do the expansion along it. This gives us:

$$\begin{aligned} \det A = -0 \begin{vmatrix} 1 & 1 & 0 \\ 2 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} + 2 \begin{vmatrix} 3 & 2 & 4 \\ 2 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} \\ - 0 \begin{vmatrix} 3 & 2 & 4 \\ 1 & 1 & 0 \\ 7 & 5 & 3 \end{vmatrix} + 0 \begin{vmatrix} 3 & 2 & 4 \\ 1 & 1 & 0 \\ 2 & 1 & 0 \end{vmatrix} \\ = 2 \cdot \left(4 \cdot \begin{vmatrix} 2 & 1 \\ 7 & 5 \end{vmatrix} + 3 \cdot \begin{vmatrix} 3 & 2 \\ 2 & 1 \end{vmatrix} \right) \\ = 2 \cdot (4 \cdot 3 + 3 \cdot (-1)) = 18 \end{aligned}$$

This is the same result as in example 6.3.3, but with less work, since we only needed to find one 3×3-determinant.

Determinants and row operations

When doing row operations on a matrix, we get a new matrix. This matrix doesn't necessarily have the same determinant as the original. However, it turns out that the determinant changes in controlled ways under row operations. We can exploit this to ease our calculations.

Theorem 6.3.2

Let A be a $n \times n$-matrix, and let B be matrix we get by doing a row operation on A. Then we have the following correspondance between the determinant of A

and B, based on which type of row operation done:

Row operation	Result
Multiplication by a number k	$\det B = k \cdot \det A$
Add a multiple of a row to another	$\det B = \det A$
exchange two rows	$\det B = -\det A$

Example 6.3.5

We find $\det A$, where

$$A = \begin{bmatrix} 3 & 3 & 12 \\ 2 & 2 & 13 \\ 4 & 2 & 19 \end{bmatrix}$$

$$\begin{aligned} \det A = \begin{vmatrix} 3 & 3 & 12 \\ 2 & 2 & 13 \\ 4 & 2 & 19 \end{vmatrix} &= 3 \cdot \begin{vmatrix} 1 & 1 & 4 \\ 2 & 2 & 13 \\ 4 & 2 & 19 \end{vmatrix} \\ &= 3 \cdot \begin{vmatrix} 1 & 1 & 4 \\ 0 & 0 & 5 \\ 0 & -2 & 3 \end{vmatrix} = -3 \cdot \begin{vmatrix} 1 & 1 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{vmatrix} \\ &= -3 \cdot 1 \cdot \begin{vmatrix} -2 & 3 \\ 0 & 5 \end{vmatrix} \\ &= -3 \cdot 1 \cdot (-2) \cdot 5 = 30 \end{aligned}$$

Here we started by doing row operations on the matrix, while simultaneously keeping track of how the determinant changed.

First we multiplied the first row with $1/3$. This made the determinant of the new matrix $1/3$ the size of the determinant of A, so we had to multiply it with 3 for the numbers to stay equal

After that we subtracted a multiple of the first row from the two other rows, which resulted in no change of the determinant.

Then we exchanged the two bottom rows, which made the determinant change sign.

At the end we expanded along the first column.

Triangulære matriser

We say that a $n \times n$-matrix is **upper triangular** if all the numbers below the diagonal are 0, that is:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ 0 & a_{22} & a_{23} & \cdots & a_{2n} \\ 0 & 0 & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & a_{nn} \end{bmatrix}$$

Similarly, a $n \times n$-matrix is **lower triangular** if all numbers above the diagonal are 0, that is:

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ a_{21} & a_{22} & 0 & \cdots & 0 \\ a_{31} & a_{32} & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & a_{n3} & \cdots & a_{nn} \end{bmatrix}$$

Example 6.3.6

In example 6.3.5 we used row operations to get our matrix on the form:

$$\begin{bmatrix} 1 & 1 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{bmatrix}$$

This matrix is upper triangular.

Theorem 6.3.3

Let A be a (upper or lower) triangular $n \times n$-matrix. Then the determinant of A is equal to the product of the numers along the diagonal of A:

$$\det A = a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn}$$

Example 6.3.7

$$\begin{vmatrix} 1 & 1 & 4 \\ 0 & -2 & 3 \\ 0 & 0 & 5 \end{vmatrix} = 1 \cdot (-2) \cdot 5$$

A good strategy when calculating the determinant of a big and unpleasant matrix, is to reduce it to an upper triangular form (and keep track of how the determinant changes with the help of Theorem 6.3.2), and then calculate the determinant of the triangular matrix using Theorem 6.3.3.

Some nice results on calculation

Theorem 6.3.4

The determinant of a product of two matrices is equal to the product of the determinants. That is: If A and B are two $n \times n$-matrices, then

$$\det(AB) = (\det A)(\det B).$$

Theorem 6.3.5

The determinant do not change under transposition. That is: If A is a $n \times n$-matrix, then

$$\det A = \det A^T.$$

Karakterisering av inverterbarhet

We can also prove that a square matrix is invertible if and only if it has a nonzero determinant, thus we can expand Theorem 6.2.4:

Theorem 6.3.6

Let A be a $n \times n$-matrix. The following statements are equivalent

1. A is invertible
2. $\det A \neq 0$
3. The columns of A are linearly independent
4. The rows of A are linearly independent
5. $\text{rank} A = n$

6. $A\mathbf{x} = \mathbf{b}$ has a unique solution $(\mathbf{x} = A^{-1}\mathbf{b})$
7. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{c}$.
8. The reduced row echelon form of A is I_n
9. A has a pivot element in every row
10. $A\mathbf{x} = \mathbf{0}$ has only one solution, $\mathbf{x} = \mathbf{0}$
11. $\text{Col}(A) = \mathbb{R}^n$
12. $\text{Row}(A) = \mathbb{R}^n$

Problem Set - Linear Indepence and Determinant

1. Given the vectors

$$\mathbf{x} = \begin{bmatrix}1\\0\end{bmatrix}, \mathbf{y} = \begin{bmatrix}1\\1\end{bmatrix}, \mathbf{v} = \begin{bmatrix}2\\2\end{bmatrix}, \mathbf{w} = \begin{bmatrix}-1\\0\end{bmatrix},$$

determine which of the following sets are linearly independent

a) $\{\mathbf{x}, \mathbf{v}\}$

b) $\{\mathbf{x}, \mathbf{w}\}$

c) $\{\mathbf{y}, \mathbf{v}\}$

d) $\{\mathbf{y}, \mathbf{w}\}$

e) $\{\mathbf{x}, \mathbf{y}, \mathbf{v}\}$

2. Given the vectors

$$\mathbf{x} = \begin{bmatrix}1\\2\\0\end{bmatrix}, \quad \mathbf{y} = \begin{bmatrix}0\\2\\2\end{bmatrix}, \text{ and } \mathbf{v} = \begin{bmatrix}1\\2\\3\end{bmatrix}$$

a) Are $\{\mathbf{x}, \mathbf{y}, \mathbf{v}\}$ a linearly independent set?

b) Is the vector

$$\mathbf{w} = \begin{bmatrix}4\\5\\6\end{bmatrix}$$

in the subspace span$\{\mathbf{x}, \mathbf{y}, \mathbf{v}\}$?

c) Solve, if possible, the vector equation

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{v} = \mathbf{w}$$

3. Find the Column space, Row Space and Null space of the following matrices

a)

$$\begin{bmatrix}1 & 5 & 0 & 0\\0 & 0 & 0 & 1\end{bmatrix}$$

b)

$$\begin{bmatrix}1 & 0\\0 & 1\\0 & -1\end{bmatrix}$$

c)

$$\begin{bmatrix}0 & 2 & 1\\0 & 0 & 4\\0 & 0 & 0\end{bmatrix}$$

d)

$$\begin{bmatrix}0 & 1 & 5\\2 & 3 & -1\\-8 & 0 & 2\end{bmatrix}$$

e)

$$\begin{bmatrix}1 & 2 & 5\\0 & 0 & 3\end{bmatrix}$$

4. Determine the determinant of the following matrices

a)

$$\begin{bmatrix}1 & 0\\0 & 2\end{bmatrix}$$

b)

$$\begin{bmatrix}2 & 2 & 4\\0 & 1 & 3\\0 & 0 & 3\end{bmatrix}$$

c)

$$\begin{bmatrix}0 & 0 & 0 & 0\\1 & 2 & 3 & 4\\5 & 6 & 7 & 8\\10 & 1 & 2 & 3\end{bmatrix}$$

d)

$$\begin{bmatrix}1 & 0 & 4 & 2 & 2\\2 & 0 & 2 & 0 & 3\\0 & 1 & 3 & 1 & 0\\0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 1 & 0\end{bmatrix}$$

e)

$$\begin{bmatrix} 5 & 9 \\ 2 & 3 \end{bmatrix}$$

5. Is the following matrices invertible?

a)

$$\begin{bmatrix} 1 & 7 \\ -1 & 1 \end{bmatrix}$$

b)

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

6. Let A be the matrix

$$\begin{bmatrix} a & b & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & y & z \end{bmatrix}$$

a) Find an expression of $\det(A)$ with a, b, c, x, y, z as variables.

b) Determine for which values

$$a, b, c, x, y, z$$

the matrix A is invertible.

Solutions - Linear Indepence and Determinant

A set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$ is linearly independent if the sum

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_t\mathbf{v}_t$$

equals the zero vector only when $a_1, a_2, \ldots, a_t$ are all zero. This is equal to

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_t \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{bmatrix} = \mathbf{0}$$

only having the trivial solution, i.e.

$$\begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_t \end{bmatrix} = \mathbf{0}.$$

Thus, we can check linearly independence of a set $\{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_t\}$, by reducing the matrix

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_t \end{bmatrix}$$

and check if all columns have a pivot-element. We also remember that for quadratic matrices, $n \times n$, every column has a pivot element if and only if the determinant is nonzero. Hence, if we have n vectors in $\mathbb{R}^n$, we can simply check the determinant of

$$\begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_t \end{bmatrix}.$$

1.

a)

$$\begin{bmatrix}\mathbf{x} & \mathbf{v}\end{bmatrix} = \begin{bmatrix}1 & 2\\0 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \implies \text{linearly independent.}$$

or

$$\begin{vmatrix}\mathbf{x} & \mathbf{v}\end{vmatrix} = \begin{vmatrix}1 & 2\\0 & 2\end{vmatrix} = 1\cdot 2 - 0\cdot 2 = 2 \neq 0 \implies \text{linearly independent.}$$

b)

$$\begin{bmatrix}\mathbf{x} & \mathbf{w}\end{bmatrix} = \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \sim \begin{bmatrix}1 & -1\\0 & 0\end{bmatrix} \implies \text{linearly dependent.}$$

or

$$\begin{vmatrix}\mathbf{x} & \mathbf{w}\end{vmatrix} = \begin{vmatrix}1 & -1\\0 & 0\end{vmatrix} = 1\cdot 0 - 0\cdot(-1) = 0 \implies \text{linearly dependent.}$$

c)

$$\begin{bmatrix}\mathbf{y} & \mathbf{v}\end{bmatrix} = \begin{bmatrix}1 & 2\\1 & 2\end{bmatrix} \sim \begin{bmatrix}1 & 2\\0 & 0\end{bmatrix} \implies \text{linearly dependent.}$$

or

$$\begin{vmatrix}\mathbf{y} & \mathbf{v}\end{vmatrix} = \begin{vmatrix}1 & 2\\1 & 2\end{vmatrix} = 1\cdot 2 - 1\cdot 2 = 0 \implies \text{linearly dependent.}$$

d) $\{\mathbf{y}, \mathbf{w}\}$

$$\begin{bmatrix}\mathbf{y} & \mathbf{w}\end{bmatrix} = \begin{bmatrix}1 & -1\\1 & 0\end{bmatrix} \sim \begin{bmatrix}1 & 0\\0 & 1\end{bmatrix} \implies \text{linearly independent.}$$

or

$$\begin{vmatrix}\mathbf{y} & \mathbf{w}\end{vmatrix} = \begin{vmatrix}1 & -1\\1 & 0\end{vmatrix} = 1\cdot 0 - 1\cdot(-1) = -1 \neq 0 \implies \text{linearly independent.}$$

e)

$$\begin{bmatrix}\mathbf{x} & \mathbf{y} & \mathbf{v}\end{bmatrix} = \begin{bmatrix}1 & 1 & 2\\0 & 1 & 2\end{bmatrix}$$

We have more columns than rows, and since the number of rows is the maximal amount of pivot elements, we can't have a pivot element in each column. Thus, the set is linearly independent.

Theorem 6.3.7

If we have vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ in $\mathbb{R}^m$, and $m < n$, then the vectors are linearly dependent.

We can also state this as

Remark 6.3.1

If U is a subspace of V, and V has dimension m, then

$$\dim(U) \leq m.$$

2.

a)

$$\begin{aligned}
\begin{vmatrix} \mathbf{x} & \mathbf{y} & \mathbf{v} \end{vmatrix} &= \begin{vmatrix} 1 & 0 & 1 \\ 2 & 2 & 2 \\ 0 & 2 & 3 \end{vmatrix} \\
&= 1 \cdot \begin{vmatrix} 2 & 2 \\ 2 & 3 \end{vmatrix} - 0 \cdot \begin{vmatrix} 2 & 2 \\ 0 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 2 & 2 \\ 0 & 2 \end{vmatrix} \\
&= 1 \cdot (2 \cdot 3 - 2 \cdot 2) - 0 \cdot (2 \cdot 3 - 0 \cdot 2) + 1 \cdot (2 \cdot 2 - 0 \cdot 2) \\
&= 2 + 0 + 4 = 6 \neq 0
\end{aligned}$$

The determinant is nonzero, so the vectors are linearly independent.

b) We see that $U = \text{span}(\mathbf{x}, \mathbf{y}, \mathbf{v})$ is given by all possible sums of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{v}$, also since these vectors are linearly independent we know they form a basis for the subspace U. Thus the dimension of U is 3, and since U lie as a subspace of $\mathbb{R}^3$, we have $\mathbb{R}^3 = U$. Now,

$$\mathbf{w} \in \mathbb{R}^3 = U$$

so $\mathbf{w}$ is in u.

Alternatively, we could observe that for $\mathbf{w}$ to lie in U, there must exist values a, b, c such that

$$a\mathbf{x} + b\mathbf{y} + c\mathbf{v} = \mathbf{w}$$

or equivalently, that the equation

$$\begin{bmatrix} \mathbf{x} & \mathbf{y} & \mathbf{v} \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \mathbf{w}$$

has a solution. Now, we use that the determinant of the matrix is non-zero, so there exists a unique solution of this equation. Hence $\mathbf{w}$ lie in U.

c) We construct the augmented matrix and reduce

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 2 & 2 & 2 & 5 \\ 0 & 2 & 3 & 6 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & -3 \\ 0 & 2 & 3 & 6 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 2 & 0 & -3 \\ 0 & 0 & 3 & 9 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 4 \\ 0 & 0 & 0 & \frac{-3}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -\frac{3}{2} \\ 0 & 0 & 1 & 3 \end{array}\right]$$

so the solution is

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ -\frac{3}{2} \\ 3 \end{bmatrix}$$

and

$$\mathbf{w} = \mathbf{x} - \frac{3}{2}\mathbf{y} + 3\mathbf{v}$$

3.

a) This matrix is already in reduced row echelon form, so we can read out at once that the basis of the column space is

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

i.e. $\mathrm{Col}(A) = \mathbb{R}^2$. The basis of the row space is

$$\begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

To find the basis of the null space, we choose x_2 and x_3 as our free variables and get the general solution of the homogeneous equation as

$$\begin{array}{llllll} x_1+ & 5x_2 & & & = 0 \\ & x_2 & & & = t \\ & & x_3 & & = s \\ & & & x_4 & = 0 \end{array} \implies \begin{aligned} x_1 &= -5t \\ x_2 &= t \\ x_3 &= s \\ x_4 &= 0 \end{aligned}$$

or on vector form

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

and a basis for the null space is

$$\begin{bmatrix} -5 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}$$

b) Basis Col(A):

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$$

Basis Row(A):

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Basis Null(A):

$$\emptyset$$

c)

$$\begin{bmatrix} 0 & 2 & 1 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Basis Col(A):

$$\begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 4 \\ 0 \end{bmatrix}$$

Basis Row(A):

$$\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix}$$

Basis Null(A):

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

d)

$$\begin{bmatrix} 0 & 1 & 5 \\ 2 & 3 & -1 \\ -8 & 0 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 5 \\ -8 & 0 & 2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 5 \\ 0 & 12 & -2 \end{bmatrix}$$
$$\sim \begin{bmatrix} 2 & 3 & -1 \\ 0 & 1 & 5 \\ 0 & 0 & -62 \end{bmatrix}$$
$$\sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Basis Col(A):

$$\begin{bmatrix} 0 \\ 2 \\ -8 \end{bmatrix}, \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ -1 \\ 2 \end{bmatrix}$$

Alternatively, since the matrix has max rank, we could have chosen the basis

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis Row(A):

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

Basis Null(A):

$$\emptyset$$

e)

$$\begin{bmatrix} 1 & 2 & 5 \\ 0 & 0 & 3 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Basis Col(A):

$$\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Basis Row(A):

$$\begin{bmatrix}1\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Basis Null(A):

$$\begin{bmatrix}-2\\1\\0\end{bmatrix}$$

4.

a)

$$\begin{vmatrix}1 & 0\\0 & 2\end{vmatrix} = 1 \cdot 2$$

b)

$$\begin{vmatrix}2 & 2 & 4\\0 & 1 & 3\\0 & 0 & 3\end{vmatrix} = 2 \cdot 1 \cdot 3$$

c)

$$\begin{aligned}\begin{vmatrix}0 & 0 & 0 & 0\\1 & 2 & 3 & 4\\5 & 6 & 7 & 8\\10 & 1 & 2 & 3\end{vmatrix} &= 0 \cdot \begin{vmatrix}2 & 3 & 4\\6 & 7 & 8\\1 & 2 & 3\end{vmatrix} - 0 \cdot \begin{vmatrix}1 & 3 & 4\\5 & 7 & 8\\10 & 2 & 3\end{vmatrix} + 0 \cdot \begin{vmatrix}1 & 2 & 4\\5 & 6 & 8\\10 & 1 & 3\end{vmatrix} - 0 \cdot \begin{vmatrix}1 & 2 & 3\\5 & 6 & 7\\10 & 1 & 2\end{vmatrix}\\ &= 0 - 0 + 0 - 0 = 0\end{aligned}$$

d)

$$\begin{vmatrix}1 & 0 & 4 & 2 & 2\\2 & 0 & 2 & 0 & 3\\0 & 1 & 3 & 1 & 0\\0 & 0 & 1 & 0 & 1\\0 & 0 & 0 & 1 & 0\end{vmatrix}$$

We start by choosing row 5, and by the checker pattern

$$\begin{matrix}+ & - & + & - & +\\- & + & - & + & -\\+ & - & + & - & +\\- & + & - & + & -\\+ & - & + & - & +\end{matrix}$$

we get first that

$$\begin{vmatrix} 1 & 0 & 4 & 2 & 2 \\ 2 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 0 & 4 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix}$$

We now choose the second column, and using checker pattern

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

we get

$$\begin{vmatrix} 1 & 0 & 4 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} = -1 \cdot \begin{vmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix}$$

We choose the third column, and using the checker pattern

$$\begin{matrix} + & - & + \\ - & + & - \\ + & - & + \end{matrix}$$

we get

$$\begin{aligned} \begin{vmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} &= 0 \cdot \begin{vmatrix} 4 & 2 \\ 2 & 3 \end{vmatrix} - 1 \cdot \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} + 1 \cdot \begin{vmatrix} 1 & 4 \\ 2 & 2 \end{vmatrix} \\ &= -1(-1) + 1(-6) = -5 \end{aligned}$$

And in conclusion

$$\begin{aligned} \begin{vmatrix} 1 & 0 & 4 & 2 & 2 \\ 2 & 0 & 2 & 0 & 3 \\ 0 & 1 & 3 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{vmatrix} &= -1 \cdot \begin{vmatrix} 1 & 0 & 4 & 2 \\ 2 & 0 & 2 & 3 \\ 0 & 1 & 3 & 0 \\ 0 & 0 & 1 & 1 \end{vmatrix} \\ &= (-1) \cdot (-1) \cdot \begin{vmatrix} 1 & 4 & 2 \\ 2 & 2 & 3 \\ 0 & 1 & 1 \end{vmatrix} \\ &= (-1)(-1)(-5) = -5 \end{aligned}$$

e)

$$\begin{vmatrix} 5 & 9 \\ 2 & 3 \end{vmatrix} = 15 - 18 = -3$$

5.

a)

$$\begin{vmatrix} 1 & 7 \\ -1 & 1 \end{vmatrix} = 1 + 7 = 8$$

The determinant is nonzero, so the matrix is invertible.

b)

$$\begin{vmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{vmatrix} = 1 \cdot 1 \cdot 1 = 1$$

The determinant is nonzero, so the matrix is invertible.

6. a We follow the second column:

$$\begin{aligned} \begin{vmatrix} a & b & 0 & 0 \\ c & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & y & z \end{vmatrix} &= -b \begin{vmatrix} c & 0 & 0 \\ 0 & 0 & x \\ 0 & y & z \end{vmatrix} + 0 \cdot \begin{vmatrix} a & 0 & 0 \\ 0 & 0 & x \\ 0 & y & z \end{vmatrix} - 0 \begin{vmatrix} a & 0 & 0 \\ c & 0 & 0 \\ 0 & y & z \end{vmatrix} + 0 \begin{vmatrix} a & 0 & 0 \\ c & 0 & 0 \\ 0 & 0 & x \end{vmatrix} \\ &= -b \cdot \left(c \cdot \begin{vmatrix} 0 & x \\ y & z \end{vmatrix} - 0 \cdot \begin{vmatrix} 0 & x \\ 0 & z \end{vmatrix} + 0 \cdot \begin{vmatrix} 0 & 0 \\ 0 & y \end{vmatrix} \right) \\ &- b \cdot c \cdot (-xy) = bcxy \end{aligned}$$

6. b The matrix is invertible as long as $b, \ c, \ x$ and y is nonzero. a and z can be whatever.

Chapter 7

Diagonalization

7.1 Eigenvectors

We have seen that when multiplying a matrix with a vector we often get a completely different vector as output.

$$\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \end{bmatrix} \begin{bmatrix} -1 \\ -2 \end{bmatrix} = \begin{bmatrix} -7 \\ 2 \end{bmatrix}$$

However, every matrix has a collection of vectors which output is particularly nice. These are the so called **eigenvectors** of the matrix. Look for example at

$$\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 6 \end{bmatrix} = 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

The matrix gave us the same vector, scaled by a number, 3.

Definition 7.1.1

Let A be a $n \times n$ matrix, $\mathbf{x} \in \mathbb{R}^n$ a non-zero vector and $\lambda \in \mathbb{C}$ a scalar. $\mathbf{x}$ is an **eigenvector** of A with **eigenvalue** λ, if

$$A\mathbf{x} = \lambda \mathbf{x}$$

Remark 7.1.1

Note that we need $\mathbf{x}$ to not be the zero vector for it to be an eigenvector. There is nothing special about $A\mathbf{0} = \lambda\mathbf{0}$ since $A\mathbf{0}$ is always $\mathbf{0}$.

Also note that we have that λ can be a complex number. In fact the definition of eigenvectors is also interesting if we let both the vector and matrix to be complex-valued as well, however, we will restrict ourselves to only letting the eigenvalue being possibly complex for now.

Let us observe that if we have an eigenvector $\mathbf{x}$ to a matrix A, then any multiple of $\mathbf{x}$ is still an eigenvector with the same eigenvalue.

$$A(c\mathbf{x}) = c(A\mathbf{x}) = c(\lambda\mathbf{x}) = \lambda(c\mathbf{x})$$

Hence, it is really easy to find $A^n\mathbf{x}$ as long as $\mathbf{x}$ is an eigenvector of A.

$$\begin{aligned} A^n\mathbf{x} = A^{n-1}(A\mathbf{x}) &= A^{n-1}(\lambda\mathbf{x}) \\ &= \lambda(A^{n-1}\mathbf{x}) = \lambda(A^{n-1}(A\mathbf{x})) = \lambda(A^{n-2}(\lambda\mathbf{x})) \\ &= \lambda^2(A^{n-2}\mathbf{x}) = \cdots \\ &\cdots = \lambda^n\mathbf{x} \end{aligned}$$

Example 7.1.1

Let A be the matrix

$$\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$$

and $\mathbf{v}$ be the eigenvector

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

with eigenvalue 3. Find $A^{50}\mathbf{v}$.

$$A^{50}\mathbf{v} = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}^{50} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = 3^{50} \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

This is in fact the property we are most interested in here in this course. Let us look at how we can use this fact to find $A^n\mathbf{y}$ for any $\mathbf{y}$ as long as A has a sufficient amount of linearly independent eigenvectors.

Example 7.1.2

The matrix A given by

$$\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$$

has eigenvector $\mathbf{v}$ of eigenvalue 3 given by

$$\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

and eigenvector $\mathbf{u}$ of eigenvalue -5 given by

$$\mathbf{u} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}.$$

$\mathbf{u}$ and $\mathbf{v}$ are linearly independent (check) and any vector $\mathbf{x}$ can be written as a linear combination of them. Find $A^5\mathbf{x}$ for

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

We first want to write $\mathbf{x}$ as a linear combination of $\mathbf{u}$ and $\mathbf{v}$, that is, we want to find a and b such that

$$a\begin{bmatrix} -1 \\ 2 \end{bmatrix} + b\begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

to find a and b, we reduce the augmented matrix

$$\left[\begin{array}{cc|c} -1 & 3 & 1 \\ 2 & 2 & 0 \end{array}\right] \sim \left[\begin{array}{cc|c} 1 & 0 & -\frac{1}{4} \\ 0 & 1 & \frac{1}{4} \end{array}\right]$$

and get $a = -1/4$ and $b = 1/4$. Now,

$$\begin{aligned}
\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}^5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} &= \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}^5 \left(-\frac{1}{4} \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \frac{1}{4} \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \\
&= -\frac{1}{4} \left(\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}^5 \begin{bmatrix} -1 \\ 2 \end{bmatrix} \right) + \frac{1}{4} \left(\begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}^5 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right) \\
&= -\frac{1}{4} \cdot (-5)^5 \begin{bmatrix} -1 \\ 2 \end{bmatrix} + \frac{1}{4} \cdot 3^5 \begin{bmatrix} 3 \\ 2 \end{bmatrix} \\
&= \begin{bmatrix} -599 \\ 1684 \end{bmatrix}
\end{aligned}$$

Finding eigenvectors

Now, how do we find eigenvectors? Let us look at the defining property:

$$A\mathbf{x} = \lambda \mathbf{x}$$

In this equation we can move $\lambda\mathbf{x}$ to the left hand side and get

$$A\mathbf{x} - \lambda\mathbf{x} = \mathbf{0},$$

or equivalently

$$(A - \lambda I)\mathbf{x} = \mathbf{0}$$

where I is the identity matrix. So we are looking for non-zero solutions to $(A - \lambda)\mathbf{x} = \mathbf{0}$. By Theorem 6.3.6, we know that this is equivalent to $\det(A - \lambda I) = 0$. When writing out $\rho_A(\lambda) = \det(A - \lambda)$, we get a polynomial of order n with variable λ, which we call the **characteristic polynomial of** A.

We know therefore that the eigenvalues of A are given as the roots of $\rho_A(\lambda)$. To find the eigenvectors associated to these, we need to solve $(A - \lambda I)\mathbf{x} = \mathbf{0}$ for all the eigenvalues λ, or equivalently determining the the null space $\mathrm{Null}(A - \lambda I)$.

Theorem 7.1.1

Let A be a $n \times n$-matrix.

(a) The eigenvalues of A are the solutions λ of the equation

$$\rho_A(\lambda) = \det(A - \lambda I_n) = 0.$$

(b) If λ is an eigenvalue of A, then the associated eigenvectors are given as the non-trivial solutions of the equation

$$(A - \lambda I_n) \cdot \mathbf{x} = \mathbf{0}.$$

Example 7.1.3

Let us now use theorem 7.1.1 to find the eigenvalues and eigenvectors to the matrix

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$$

The eigenvalues are found by solving

$$\det(A - \lambda I_2) = 0,$$

Let us first see how the matrix $A - \lambda I_2$ looks like:

$$A - \lambda I_2 = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 1-\lambda & 3 \\ 4 & -3-\lambda \end{bmatrix}$$

The characteristic polynomial is

$$\begin{aligned} \rho_A(\lambda) = \det(A - \lambda I_2) &= \begin{vmatrix} 1-\lambda & 3 \\ 4 & -3-\lambda \end{vmatrix} \\ &= (1-\lambda)(-3-\lambda) - 3 \cdot 4 \\ &= \lambda^2 + 2\lambda - 15. \end{aligned}$$

This means that we can solve the second order equation

$$\lambda^2 + 2\lambda - 15 = 0$$

to find the eigenvalues. We solve it as usual and get:

$$\lambda = \frac{-2 \pm \sqrt{2^2 - 4 \cdot (-15)}}{2} = -1 \pm 4$$

We therefore have the eigenvalues: 3 and -5.

We find the eigenvectors associated to 3 by solving the equation $(A - 3I_2)\mathbf{x} = \mathbf{0}$. We solve it by reducing the matrix $(A - 3I_2)$:

$$A - 3I_2 = \begin{bmatrix} -2 & 3 \\ 4 & -6 \end{bmatrix} \sim \begin{bmatrix} -2 & 3 \\ 0 & 0 \end{bmatrix}$$

We get a free variable and the solution is

$$\mathbf{x} = t \cdot \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

for all numbers t. The eigenvectors corresponding to 3 are therefore all vectors in

$$\operatorname{span}\left\{ \begin{bmatrix} 3 \\ 2 \end{bmatrix} \right\},$$

except the zero vector.

We find all eigenvectors to -5 by solving the equation $(A + 5I_2)\mathbf{x} = \mathbf{0}$. We can solve this equation by reducing the matrix $(A + 5I_2)$:

$$A + 5I_2 = \begin{bmatrix} 6 & 3 \\ 4 & 2 \end{bmatrix} \sim \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}$$

We get a free variable, and the solution is

$$\mathbf{x} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} \cdot t$$

for all numbers t. The eigenvectors corresponding to the eigenvalue -5 are therefore all the vectors in

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ -2 \end{bmatrix} \right\},$$

except the zero vector.

Definition 7.1.2

A **diagonal matrix** is a quadratic matrix where all numbers outside of the diagonal are 0, that is a matrix on the form:

$$\begin{bmatrix} a_{11} & 0 & 0 & \cdots & 0 \\ 0 & a_{22} & 0 & \cdots & 0 \\ 0 & 0 & a_{33} & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \vdots & a_{nn} \end{bmatrix}$$

The eigenvalues of a diagonal matrix are easily obtained

Theorem 7.1.2

The eigenvalues of a diagonal matrix are the numbers along the diagonal.

Proof. The characteristic polynomial of the diagonal matrix A is

$$\det(A - \lambda \cdot I_n) = (a_{11} - \lambda)(a_{22} - \lambda) \cdots (a_{nn} - \lambda).$$

So the numbers $a_{11}, \ldots, a_{nn}$ are the solutions of the equation $\det(A - \lambda \cdot I_n) = 0$. $\square$

We can also note that the eigenvectors gives yet another condition for the invertability of a matrix

Theorem 7.1.3

A $n \times n$-matrix A has 0 as an eigenvalue if and only if it is not invertible.

We can add this to our running theorem 6.3.6 and thus have

Theorem 7.1.4

Let A be a $n \times n$-matrix. The following statements are equivalent

1. A is invertible
2. $\det A \neq 0$
3. A do not have 0 as an eigenvalue
4. The columns of A are linearly independent
5. The rows of A are linearly independent
6. $\mathrm{rank} A = n$
7. $A\mathbf{x} = \mathbf{b}$ has a unique solution $(\mathbf{x} = A^{-1}\mathbf{b})$
8. $A\mathbf{x} = \mathbf{b}$ has a solution for every $\mathbf{c}$.
9. The reduced row echelon form of A is I_n
10. A has a pivot element in every row
11. $A\mathbf{x} = \mathbf{0}$ has only one solution, $\mathbf{x} = \mathbf{0}$
12. $\mathrm{Col}(A) = \mathbb{R}^n$

13. $\mathrm{Row}(A) = \mathbb{R}^n$

A last useful result we can note is that the eigenvectors corresponding to different eigenvalues are linearly independent:

Theorem 7.1.5

Let A be a $n \times n$ matrix. Let $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_t$ be eigenvectors of A associated to *different* eigenvalues λ_1, λ_2, ..., λ_t. Then the vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ..., $\mathbf{v}_t$ are linearly independent.

Example 7.1.4

We find the eigenvalues of

$$A = \begin{bmatrix} -8 & 0 & 6 \\ 12 & 4 & -6 \\ -20 & 0 & 14 \end{bmatrix},$$

and the corresponding eigenvectors.

The characteristic polynomial of A is:

$$\begin{aligned} \det(A - \lambda I_3) &= \begin{vmatrix} -8-\lambda & 0 & 6 \\ 12 & 4-\lambda & -6 \\ -20 & 0 & 14-\lambda \end{vmatrix} \\ &= (4-\lambda) \cdot \begin{vmatrix} -8-\lambda & 6 \\ -20 & 14-\lambda \end{vmatrix} \\ &= (4-\lambda)\big((-8-\lambda)(14-\lambda) + 6 \cdot 20\big) \\ &= (4-\lambda)(\lambda^2 - 6\lambda + 8) \end{aligned}$$

We find the eigenvalues of A by solving the third degree equation

$$(4-\lambda)(\lambda^2 - 6\lambda + 8) = 0.$$

This equation is equivalent to

$$4 - \lambda = 0 \qquad \text{or} \qquad \lambda^2 - 6\lambda + 8 = 0.$$

The second degree equation $\lambda^2 - 6\lambda + 8 = 0$ have solutions

$$\lambda = \frac{6 \pm \sqrt{6^2 - 4 \cdot 8}}{2} = 3 \pm 1,$$

so we have two eigenvalues: 2 og 4.

We find the eigenvectors by solving the equations

$$(A - 2I_3)\mathbf{x} = \mathbf{0} \qquad \text{og} \qquad (A - 4I_3)\mathbf{x} = \mathbf{0}.$$

For $\lambda = 2$ we reduce the matrix $A - 2I_3$:

$$\begin{bmatrix} -8-2 & 0 & 6 \\ 12 & 4-2 & -6 \\ -20 & 0 & 14-2 \end{bmatrix} \sim \begin{bmatrix} -10 & 0 & 6 \\ 12 & 2 & -6 \\ -20 & 0 & 12 \end{bmatrix}$$
$$\sim \begin{bmatrix} -10 & 0 & 6 \\ 12 & 2 & -6 \\ 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} -5 & 0 & 3 \\ 6 & 1 & -3 \\ 0 & 0 & 0 \end{bmatrix}.$$

The first row gives the equation $-5x_1 + 3x_3 = 0$. This is satisfied for example when $x_1 = 3$ and $x_3 = 5$. These two values gives in the second equation that $x_2 = -3$. The result is that the eigenvectors of 2 are the non-zero vectors in

$$\operatorname{span}\left\{ \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix} \right\}.$$

To find the eigenvectors of 4 we reduce $A - 4I_3$:

$$\begin{bmatrix} -8-4 & 0 & 6 \\ 12 & 4-4 & -6 \\ -20 & 0 & 14-4 \end{bmatrix} \sim \begin{bmatrix} -12 & 0 & 6 \\ 12 & 0 & -6 \\ -20 & 0 & 10 \end{bmatrix}$$
$$\sim \begin{bmatrix} -2 & 0 & 1 \\ 2 & 0 & -1 \\ -2 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} -2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Here we have that both x_2 and x_3 are free variables. If we choose $x_2 = 1$ and $x_3 = 0$, we get $x_1 = 0$. If we choose $x_3 = 2$, we have from the first row that

$x_1 = 1$. In this case x_2 can be chosen freely, so we can choose $x_2 = 0$. This gives us that the eigenvectors of 4 are all the non-zero vectors of

$$\operatorname{span}\left\{\begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\0\\2\end{bmatrix}\right\}.$$

Remark 7.1.2

In an effort to declutter our arguments we often use the notion of **eigenspaces** of a matrix. If λ is an eigenvalue of A, then the eigenspace, E_λ, of λ is exactly the null space we have used to find the eigenvectors:

$$E_\lambda = \operatorname{Null}(A - \lambda I)$$

The dimension of an eigenspace E_λ is called the **geometric multiplicity** of λ.

The power of $(\lambda - \lambda_i)$ in the characteristic polynomial $\rho_A(\lambda)$ of a matrix A, is called the **algebraic multiplicity** of λ, i.e. if

$$\rho_A(\lambda) = (\lambda - 1)^2(\lambda - 4)^3(\lambda + 1)$$

then the eigenvalue 1 has algebraic multiplicity 2, the eigenvalue 4 has algebraic multiplicity 3 and the eigenvalue -1 has algebraic multiplicity 1.

7.2 Diagonalization

Consider the diagonal matrix

$$D = \begin{bmatrix}3 & 0\\0 & -5\end{bmatrix}.$$

It is easy to multiply D with vectors in $\mathbb{R}^2$ or $\mathbb{C}^2$. But, it is also easy to multiply D with itself for example D^5 can be found as

$$D^5 = D \cdot D \cdot D \cdot D \cdot D = \begin{bmatrix}3^5 & 0\\0 & (-5)^5\end{bmatrix} = \begin{bmatrix}243 & 0\\0 & -3125\end{bmatrix}.$$

If we try the same with A, that is to find A^5 for

$$A = \begin{bmatrix}1 & 3\\4 & -3\end{bmatrix},$$

the calculations are more cumbersome.

Now, recall that we found that A have eigenvalues $\lambda_1 = 3$ and $\lambda_2 = -5$ with eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

This means that we have

$$A\mathbf{v}_1 = 3\mathbf{v}_1 \text{ og } A\mathbf{v}_2 = (-5)\mathbf{v}_2.$$

Rewriting these two equations by having the eigenvalues as the diagonal in a matrix D and the eigenvectors as the columns in a matrix P we get:

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \text{ og } P = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix},$$

such that

$$AP = PD.$$

Observe that P is invertable with inverse

$$P^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}.$$

We can therefore multiply both sides of $AP = PD$ with P^{-1} and get

$$A = PDP^{-1}.$$

Now we can calculate A^k for all k with the formula

$$\begin{aligned} A^k &= \overbrace{(PDP^{-1})(PDP^{-1})\cdots(PDP^{-1})}^{n \text{ times}} \\ &= PD\overbrace{(P^{-1}P)}^{=I_2}D\overbrace{(P^{-1}P)}^{=I_2}\cdots D\overbrace{(P^{-1}P)}^{=I_2}DP^{-1} \\ &= PD^kP^{-1} \end{aligned}.$$

Definition 7.2.1

A $n \times n$-matrix A is **diagonalizable** if there exists a diagonal matrix D and an invertible matrix P such that

$$A = PDP^{-1}.$$

We say that P diagonalize A in these cases.

Not all matrices are diagonalizable. We need a method to check whether we can diagonalize A.

Theorem 7.2.1

A $n \times n$-matrix A is diagonalizable if and only if A has n linearly independent eigenvectors.

Proof. We only show that enough eigenvectors implies diagonalizibility. Assume A has n linearly independent eigenvectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$ and corresponding eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$. For every eigenvector we have

$$A\mathbf{v}_k = \lambda_k \mathbf{v}_k.$$

As above, we can organize these n equations into a matrix equation

$$AP = PD,$$

where

$$D = \begin{bmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

and

$$P = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 & \cdots & \mathbf{v}_n \end{bmatrix}.$$

Since $\mathbf{v}_1$, $\mathbf{v}_2 \ldots \mathbf{v}_n$ are linearly independent, the $n \times n$-matrix P is invetable. We can therefore find an inverse P^{-1} and we have

$$A = PDP^{-1}.$$

We conclude that A is diagonalizable.

□

Remark 7.2.1

Generally, if a $n \times n$-matrix has n different eigenvalues, then it is diagonalizable since each eigenvalue has at least one eigenvector and we know that eigenvectors belonging to different eigenvalues are linearly independent.

Example 7.2.1

We have already looked at the matrix

$$A = \begin{bmatrix} -8 & 0 & 6 \\ 12 & 4 & -6 \\ -20 & 0 & 14 \end{bmatrix},$$

and know that the eigenvalues to A are 2 and 4. The eigenvectors of 2 are the non-zero vectors in

$$\operatorname{span}\left\{ \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix} \right\}.$$

The eigenvectors of 4 are the non-zero vectors in

$$\operatorname{span}\left\{ \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \right\}.$$

We observe that we have three linearly independent eigenvectors

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -3 \\ 5 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \text{ og } \mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

This means that A is diagonalizable with the diagonal matrix

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

and invertable matrix

$$P = \begin{bmatrix} 3 & 0 & 1 \\ -3 & 1 & 0 \\ 5 & 0 & 2 \end{bmatrix}.$$

Example 7.2.2

The matrix

$$\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

have eigenvalues

$$\lambda = \pm i$$

with the nonzero vectors in $\operatorname{span}\left\{\begin{bmatrix} i \\ 1 \end{bmatrix}\right\}$ as eigenvectors to i, and the nonzero vectors in $\operatorname{span}\left\{\begin{bmatrix} 1 \\ i \end{bmatrix}\right\}$ as eigenvectors to $-i$.

This means that A is diagonalizable as a *complex* matrix with diagonal matrix

$$D = \begin{bmatrix} i & 0 \\ 0 & -i \end{bmatrix}$$

and invertable matrix

$$P = \begin{bmatrix} i & 1 \\ 1 & i \end{bmatrix}.$$

Example 7.2.3

We consider the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}.$$

The eigenvalue of A are 1:

$$\det\left(\begin{bmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{bmatrix}\right) = (1-\lambda)^2.$$

The eigenvectors of 1 are the non-zero vectors of the zero space to the matrix

$$A - I_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

This is the vectors spanned by $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. We can't construct two linearly independent vectors from only one, so in particular we see that A do not have 2 linearly independent eigenvectors and is therefore not diagonalizable.

We can also formulate our findings on diagonalizability in terms of the algebraic and geometric multiplicity of the eigenvalues:

Theorem 7.2.2

Let A be a $n \times n$-matrix. If the algebraic multiplicity and geometric multiplicity of each eigenvalue of A is equal, then A is diagonalizable.

Proof. Let

$$\rho_A(\lambda) = a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0$$

be the characteristic polynomial of A. It has degree n since A is a $n \times n$-matrix. Now, by the fundamental theorem of algebra we can decompose $\rho_A(\lambda)$ as follows

$$\rho_A(\lambda) = A(\lambda - \lambda_1)^{\alpha_1}(\lambda - \lambda_2)^{\alpha_2} \cdots (\lambda - \lambda_t)^{\alpha_t}$$

where A is some constant, $\lambda_1, \dots, \lambda_t \in \mathbb{C}$ are eigenvalues (potentially complex) of A, and $\alpha_1 + \alpha_2 + \cdots + \alpha_t = n$.

Now, since the geometric multiplicity of λ_i equals the algebriac multiplicity α_i, we can pick a set of α_i linearly independent eigenvectors belonging to λ_i. Since we know that eigenvectors belonging to different eigenvalues are linearly independent, we can collect the sets of linearly independent eigenvectors of each eigenvalue into a big set of linearly independent eigenvectors.

Further, since $\alpha_1 + \alpha_2 + \cdots + \alpha_t = n$, we have then found n linearly independent eigenvectors of A, and we conclude that A is diagonalizable. $\square$

Symmetric matrices

A particularly nice family of matrices when it comes to diagonalizability is the real valued symmetric ones. We will see in a bit that they are always diagonalizable.

Definition 7.2.2

A real valued matrix is called **symmetric** if $A = A^T$.

Example 7.2.4

The matrix

$$\begin{bmatrix} 1 & -5 & 7 \\ -5 & 2 & -13 \\ 7 & -13 & 3 \end{bmatrix}$$

is symmetric.

Theorem 7.2.3

Let A be a symmetric $n \times n$-matrix. Then A has n real valued eigenvalues and A is diagonalizable.

7.3 Jordan form (optional)

Remark 7.3.1

The readability of this section is not tried optimized, venture on with caution.

What do we do if there aren't enough linearly independent eigenvectors to diagonalize a matrix A? In this case we would still like to construct an invertible matrix P such that

$$A = PJP^{-1}$$

for a matrix J as diagonal as possible. Such a matrix, it turns out, is a **Jordan form** matrix and is found through the construction of additional **generalized eigenvectors** which is put alongside our original eigenvectors into P.

A matrix of Jordan form can be seen as a big $n \times n$ matrix J subdivided into smaller $l_j \times l_j$ matrices J_j along the diagonal, and zeros elsewhere,

$$J = \begin{bmatrix} J_1 & & \\ & \ddots & \\ & & J_t \end{bmatrix}$$

the jordan blocks are on the form

$$J_i = \begin{bmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ & \lambda_i & 1 & 0 & \cdots & 0 \\ & & \ddots & & & \vdots \\ & & & \lambda_i & 1 & 0 \\ & & & & \lambda_i & 1 \\ & & & & & \lambda_1 \end{bmatrix}$$

with only zeros below the diagonal.

Suppose A has t linearly independent eigenvectors. Then it is can be written as PJP^{-1} for a Jordan form matrix with t blocks $J_1, \ldots, J_t$. Each block has an eigenvalue on the diagonal with 1's in the entry right above. The matrix P consists of the generalized eigenvectors of A.

Definition 7.3.1

Let A be a $n \times n$-matrix and λ an eigenvalue of A. A non-zero vector $\mathbf{x}$ is a **generalized eigenvector** of A if for some integer p, we have

$$(A - \lambda I)^p \mathbf{x} = \mathbf{0}$$

we say that $\mathbf{x}$ is a generalized eigenvector of **rank** p if also

$$(A - \lambda I)^{p-1}\mathbf{x} = \mathbf{0}$$

In particular, we have that any eigenvector of λ is a generalized eigenvector of rank 1.

Given a generalized eigenvector $\mathbf{v}$ of rank r belonging to *lambda*, we define the vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ iteratively as

$$\begin{aligned}
\mathbf{v}_r &= (A - \lambda I)^0\mathbf{v} = I\mathbf{v} = \mathbf{v}\\
\mathbf{v}_{r-1} &= (A - \lambda I)\mathbf{v}_r = (A - \lambda I)^1\mathbf{v}\\
\mathbf{v}_{r-2} &= (A - \lambda I)\mathbf{v}_{r-1} = (A - \lambda I)^2\mathbf{v}\\
&\vdots\\
\mathbf{v}_2 &= (A - \lambda I)\mathbf{v}_3 = (A - \lambda I)^{r-2}\mathbf{v}\\
\mathbf{v}_1 &= (A - \lambda I)\mathbf{v}_2 = (A - \lambda I)^{r-1}\mathbf{v}
\end{aligned}$$

Note that $\mathbf{v}_1$ is an eigenvector as $v_1 \neq \mathbf{0}$ and $(A - \lambda I)\mathbf{v}_1 = (A - \lambda I)^r\mathbf{v} = 0$. The vectors $\mathbf{v}_1, \mathbf{v}_2, \dots, \mathbf{v}_r$ form a **chain of generalized eigenvectors of length** r.

Theorem 7.3.1

The vectors in a chain of generalized eigenvectors are linearly independent.

Theorem 7.3.2

For an eigenvalue λ of algebraic multiplicity k, there exist p chains of generalized eigenvectors,

$$\begin{array}{ccccccc}
\mathbf{v}_1^1 & , & \mathbf{v}_1^2 & , & \cdots & , & \mathbf{v}_1^p\\
\mathbf{v}_2^1 & & \mathbf{v}_2^2 & & & & \mathbf{v}_2^p\\
\vdots & & \vdots & & & & \vdots\\
\vdots & & \mathbf{v}_{r_2}^2 & & & & \mathbf{v}_{r_p}^p\\
\mathbf{v}_{r_1}^1 & & & & & &
\end{array}$$

such that the collection of all these vectors

$$\{\mathbf{v}_1^1, \dots, \mathbf{v}_{r_1}^1, \mathbf{v}_1^2, \dots, \mathbf{v}_{r_2}^2, \cdots, \mathbf{v}_1^p, \dots, \mathbf{v}_{r_p}^p\}$$

forms a linearly independent set, and $\sum_i^p r_i = k$. Here r_i denotes the length of the ith chain.

Theorem 7.3.3

Given p chains, which is denoted as in theorem 7.3.2, the vectors $\{\mathbf{v}_i^j\}$ are linearly independent if and only if

$$\mathbf{v}_1^1, \mathbf{v}_1^2, \dots, \mathbf{v}_1^p$$

are linearly independent.

Also, if each chain correspond to different eigenvalues, then the vectors $\{\mathbf{v}_i^j\}$ are also linearly independent.

The two preceding theorems tells us that there exist a matrix J on jordan form, and a matrix P such that $A = PJP^{-1}$ for any $n \times n$-matrix A.

Suppose A have the eigenvalues $\lambda_1, \dots, \lambda_t$ and also that each eigenvalue λ_l has p_l chains $\{\mathbf{v}_{i,l}^j\}$ of generalized eigenvectors as in Theorem 7.3.2. Denote by $P(\lambda_l, s)$ the matrix which columns are the generalized eigenvectors of the sth chain of λ_l,

$$P(\lambda_l, s) = \begin{bmatrix} \mathbf{v}_1^s & \mathbf{v}_2^s & \cdots & \mathbf{v}_{r_s}^s \end{bmatrix}$$

where r_s is the length of the sth chain, then he matrix P is on the form

$$P = \begin{bmatrix} P(\lambda_1, 1) & \cdots & P(\lambda_1, p_1) & P(\lambda_2, 1) & \cdots & P(\lambda_2, p_2) & \cdots & P(\lambda_t, 1) & \cdots & P(\lambda_t, p_t) \end{bmatrix}$$

then J is on the form

$$J = \begin{bmatrix} J(\lambda_1,1) & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & J(\lambda_1,p_1) & & & & & & & \\ & & & J(\lambda_2,1) & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & J(\lambda_2,p_2) & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & \ddots & & \\ & & & & & & & & J(\lambda_t,1) & & \\ & & & & & & & & & \ddots & \\ & & & & & & & & & & J(\lambda_t,p_t) \end{bmatrix}$$

where the jordan block $J(\lambda_l, s)$ has size $r_s \times r_s$, whit r_s equal the length of the

sth chain of eigenvalue λ_l:

$$\overbrace{\begin{bmatrix} \lambda_i & 1 & 0 & 0 & \cdots & 0 \\ & \lambda_i & 1 & 0 & \cdots & 0 \\ & & \ddots & & & \vdots \\ & & & \lambda_i & 1 & 0 \\ & & & & \lambda_i & 1 \\ & & & & & \lambda_1 \end{bmatrix}}^{r_s} \; \updownarrow r_s$$

Example 7.3.1

Let

$$A = \begin{bmatrix} 0 & 1 \\ -4 & 4 \end{bmatrix}$$

Then the characteristic polynomial of A is $\rho_A(\lambda) = \lambda^2 - 4t + 4 = (\lambda - 2)^2$. The eigenspace

$$E_2 = \operatorname{Null} \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} = \operatorname{Null} \begin{bmatrix} 1 & -1/2 \\ 0 & 0 \end{bmatrix} = \operatorname{span} \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$$

is only one-dimensional, i.e. the geometric multiplicity of 2 is 1 which is lower than the algebraic multiplicity which is 2. The matrix is therefore not diagonalizable.

We observe however that $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is in the column space of $A - 2\lambda$, i.e. the equation

$$\begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix} \mathbf{x} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

has a solution, for example $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. Further, since

$$(A - 2I_2)\mathbf{v}_2 = \mathbf{v}_1$$

we see that

$$A\mathbf{v}_2 = \mathbf{v}_1 + 2\mathbf{v}_2 = \begin{bmatrix} \mathbf{v}_1 & \mathbf{v}_2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

combining this with

$$A\mathbf{v}_1 = 2\mathbf{v}_1 = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}$$

we have

$$A\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix} = \begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}$$

If we denote $\begin{bmatrix}\mathbf{v}_1 & \mathbf{v}_2\end{bmatrix}$ by P and observe this matrix is invertible, we have

$$A = P\begin{bmatrix}2 & 1\\0 & 2\end{bmatrix}P^{-1}$$

and observe that we have decomposed A into its Jordan form.

Problem Set - Eigenvectors and Diagonalization

1. Calculate the eigenvalues and associated eigenspaces of

a)
$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},$$

b)
$$\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix},$$

2. Calculate the eigenvalues and eigenvectors of

a)
$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

b)
$$\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

c)
$$\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$$

d)
$$\begin{bmatrix} 3 & 1 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & 3 \end{bmatrix}$$

3. Let the eigenvectors of

$$A = \begin{bmatrix} 0 & 0 & -1 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

be the columns of a 3×3-matrix P and calculate $P^{-1}AP$.

4. Find a formula for A^n and calculate A^{10} when

a)
$$A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$$

b)
$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

5. Find a 3×3-matrix which has eigenvalues 1, 2 and 3, with associated eigenvectors

$$\begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 6 \\ 5 \end{bmatrix} \quad \text{og} \quad \begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}.$$

Solutions - Eigenvectors and Diagonalization

1. a In order to find the eigenvalues of a matrix A we have to find the roots of the characteristic polynomial $p(\lambda) = \det(A - \lambda I)$, where A is a quadratic matrix and I is the identity matrix of same size.

In this problem, we get, if the matrix is denoted by A, that

$$\begin{aligned} p(\lambda) &= \det(A - \lambda I) \\ &= \det\left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) \\ &= \det\left(\begin{bmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{bmatrix}\right) \\ &= \lambda(\lambda - 1) \end{aligned}$$

The equation have the solutions $\lambda_1 = 0$ and $\lambda_2 = 1$. These are the eigenvalues of the matrix.

The eigenvectors of A are associated to the eigenvalues, and are found as the non-zero vectors $\mathbf{x}_1$ and $\mathbf{x}_2$ such that

$$A\mathbf{x}_1 = \lambda_1 \mathbf{x}_1 \text{ and } A\mathbf{x}_2 = \lambda_2 \mathbf{x}_2.$$

We observe that any vector on the form

$$\mathbf{x}_1 = \begin{bmatrix} 0 \\ a \end{bmatrix}$$

gives

$$A\mathbf{x}_1 = 0\mathbf{x}_1 = \mathbf{0}.$$

The vector $\mathbf{x}_1 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is therefore an eigenvector of λ_1.

We also observe that any vector on the form

$$\mathbf{x}_2 = \begin{bmatrix} a \\ 0 \end{bmatrix}$$

gives

$$A\mathbf{x}_2 = 1\mathbf{x}_2 = \mathbf{x}_2.$$

The vector $\mathbf{x}_2 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is therefore an eigenvector of λ_2.

The eigenspace associates to λ_1 is therefore the y-axis, and the eigenspace of λ_2 is the x-axis.

1.b We find the characteristic polynomial:

$$\begin{aligned}
p(\lambda) &= \det\left(\frac{1}{2}\begin{bmatrix}1 & 1\\ 1 & 1\end{bmatrix} - \begin{bmatrix}\lambda & 0\\ 0 & \lambda\end{bmatrix}\right)\\
&= \det\left(\frac{1}{2}\begin{bmatrix}1-2\lambda & 1\\ 1 & 1-2\lambda\end{bmatrix}\right)\\
&= \frac{1}{2}((1-2\lambda)(1-2\lambda)-1)\\
&= 2\lambda(\lambda-1)
\end{aligned}$$

The equation $p(\lambda) = 0$ have the solutions $\lambda_1 = 0$ and $\lambda_2 = 1$. These are the eigenvalues of the matrix.

We observe that $\mathbf{x}_1 = \begin{bmatrix}-1\\ 1\end{bmatrix}$ solves

$$A\mathbf{x}_1 = 0\mathbf{x}_1 = \mathbf{0}.$$

The vector $\mathbf{x}_1 = \begin{bmatrix}-1\\ 1\end{bmatrix}$ is therefore an eigenvector associated to $\lambda_1 = 0$.

$\mathbf{x}_2$ must lie in the null space of

$$A - I = \frac{1}{2}\begin{bmatrix}-1 & 1\\ 1 & -1\end{bmatrix}.$$

This is spanned by the vector $\begin{bmatrix}1\\ 1\end{bmatrix}$, so we choose

$$\mathbf{x}_2 = \begin{bmatrix}1\\ 1\end{bmatrix}.$$

And verify that $A\mathbf{x}_2 = 1\mathbf{x}_2 = \mathbf{x}_2$. In this case, the eigenspace of λ_1 is the line spanned by $\begin{bmatrix}-1\\ 1\end{bmatrix}$, and the eigenspace of λ_2 is the line spanned by $\begin{bmatrix}1\\ 1\end{bmatrix}$.

2. a Eigenvalues are $\lambda_1 = -2, \lambda_2 = i$ and $\lambda_3 = -i$.

The eigenspaces are spanned by the corresponding eigenvectors

$$\begin{bmatrix}0\\ 1\\ 0\end{bmatrix}, \begin{bmatrix}i\\ 1\\ 1\end{bmatrix}, \begin{bmatrix}-i\\ 1\\ 1\end{bmatrix}$$

2. b

$$\det(A - \lambda I) = (\lambda + 1)^2(\lambda - 2)$$

gives eigenvalues

$$\lambda_1 = -1, \quad \lambda_2 = 2$$

The eigenspaces are spanned by

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

3. We found the eigenvalues and eigenvectors of the matrix in problem 2a.

$$P = \begin{bmatrix} 0 & i & -i \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} 0 & 2 & -2 \\ -i & 0 & 1 \\ i & 0 & 1 \end{bmatrix}$$

Calcualted

$$P^{-1}AP = \begin{bmatrix} -2 & 0 & 0 \\ 0 & i & 0 \\ 0 & 0 & -i \end{bmatrix}$$

Which expectedly is a diagonal matrix where the elements along the diagonal are the eigenvalues of A.

4. a The eigenvalues of A is 3 and -5 with eigenvectors respectively $\begin{bmatrix} 3 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. We therefore define

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \text{ og } P = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}$$

and find

$$P^{-1} = \frac{1}{8}\begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}.$$

Now, we know $A = PDP^{-1}$, and therefore $A^k = PD^kP^{-1}$. We express this last product explicitly :

$$\begin{aligned} A^k &= \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix}\begin{bmatrix} 3^k & 0 \\ 0 & (-5)^k \end{bmatrix}\frac{1}{8}\begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix} \\ &= \frac{1}{8}\begin{bmatrix} 6 \cdot 3^k + 2 \cdot (-5)^k & 3 \cdot 3^k - 3(-5)^k \\ 4 \cdot 3^k - 4 \cdot (-5)^k & 2 \cdot 3^k + 6 \cdot (-5)^k \end{bmatrix}. \end{aligned}$$

For $k = 10$ we have

$$A^{10} = \begin{bmatrix} 2485693 & -3639966 \\ -4853288 & 7338981 \end{bmatrix}.$$

4. b A is upper triangular, so the eigenvalues are the numbers along the diagonal of A: 2, 3 and 5. We make the matrix P and D:

$$P = \begin{bmatrix} 1 & 3 & 25 \\ 0 & 1 & 15 \\ 0 & 0 & 6 \end{bmatrix}$$

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

We use row recution, and the fact that $[B|I] \sim [I|C]$, means $B^{-1} = C$ to find

$$P^{-1} = \begin{bmatrix} 1 & -3 & 10/3 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1/6 \end{bmatrix}$$

These reduction steps are given as:

$$\begin{aligned}
\begin{bmatrix} 1 & 3 & 25 & 1 & 0 & 0 \\ 0 & 1 & 15 & 0 & 1 & 0 \\ 0 & 0 & 6 & 0 & 0 & 1 \end{bmatrix} &\sim \begin{bmatrix} 1 & 3 & 0 & 1 & 0 & -\frac{25}{6} \\ 0 & 1 & 0 & 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 6 & 0 & 0 & 1 \end{bmatrix} \\
&\sim \begin{bmatrix} 1 & 0 & 0 & 1 & -3 & \frac{10}{3} \\ 0 & 1 & 0 & 0 & 1 & -\frac{5}{2} \\ 0 & 0 & 1 & 0 & 0 & \frac{1}{6} \end{bmatrix}
\end{aligned}$$

thus we have diagonalized the matrix A.

Notice that when $A = PDP^{-1}$, we have

$$\begin{aligned}
A^2 &= (PDP^{-1})(PDP^{-1}) = PD(P^{-1}P)DP^{-1} \\
&= PD^2P^{-1}, \\
A^3 &= (PD^2P^{-1})(PDP^{-1}) = PD^2(P^{-1}P)DP^{-1} \\
&= PD^3P^{-1},
\end{aligned}$$

and so on. In general:

$$A^n = PD^nP^{-1}$$

For $n = 10$:

$$\begin{aligned}
A^{10} &= PD^{10}P^{-1} \\
&= \begin{bmatrix} 1 & 3 & 25 \\ 0 & 1 & 15 \\ 0 & 0 & 6 \end{bmatrix} \begin{bmatrix} 2^{10} & 0 & 0 \\ 0 & 3^{10} & 0 \\ 0 & 0 & 5^{10} \end{bmatrix} \begin{bmatrix} 1 & -3 & 10/3 \\ 0 & 1 & -5/2 \\ 0 & 0 & 1/6 \end{bmatrix} \\
&= \begin{bmatrix} 1024 & 174075 & 40250650 \\ 0 & 59049 & 24266440 \\ 0 & 0 & 9765625 \end{bmatrix}
\end{aligned}$$

5. Construct a diagonal matrix D with the eigenvalues along the diagonal, and a matrix V with the eigenvectors as columns:

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix} \quad \text{og} \quad V = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 6 & 4 \\ 2 & 5 & 2 \end{bmatrix}$$

Then $A = VDV^{-1}$ is our wanted matrix. We find the inverse matrix of V:

$$V^{-1} = \begin{bmatrix} 8 & -1 & -2 \\ -2 & 0 & 1 \\ -3 & 1 & 0 \end{bmatrix}$$

Now we multiply the matrices and end up with:

$$A = VDV^{-1} = \begin{bmatrix} -9 & 2 & 2 \\ -36 & 9 & 6 \\ -22 & 4 & 6 \end{bmatrix}$$

Part III

System of ODEs

Chapter **8**

Systems of differential equations

Let us now step back to ODEs. In many applied problems, several quantities are varying continuously in time, and can be described through a **system of differential equations**[1]

$$\begin{aligned} y_1'(t) &= a_{11}y_1(t) + a_{12}y_2(t) + \cdots + a_{1n}y_n(t) \\ y_2'(t) &= a_{21}y_1(t) + a_{22}y_2(t) + \cdots + a_{2n}y_n(t) \\ &\vdots \\ y_n'(t) &= a_{n1}y_1(t) + a_{n2}y_2(t) + \cdots + a_{nn}y_n(t) \end{aligned}$$

Here $y_1, \dots, y_n$ are differentiable functions of t, with derivatives $y_1', \dots, y_n'$. The a_{ij} are constants. In our discussion we will assume that the constants a_{ij} are real numbers. Alternatively, we can write this as

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \\ \vdots \\ y_n'(t) \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \\ \vdots \\ y_n(t) \end{bmatrix}.$$

We also use the short hand

$$\mathbf{y}' = A\mathbf{y}.$$

A solution of this is a vector of functions that satisfies the equation above. Both derivation and matrix multiplication is linear, so we rediscover the superposition principle we already saw in the case of a single ODE. That is:

[1]Note that we have implicitly assumed in the following that our coefficients are constants and that the system is homogeneous. This is however not the case in general.

Theorem 8.0.1 Superposition principle

If $\mathbf{y}_1$ and $\mathbf{y}_2$ are solutions of the system

$$\mathbf{y}' = A\mathbf{y}$$

then also $c_1\mathbf{y}_1 + c_2\mathbf{y}_2$ is a solution for all real numbers c_1 and c_2.

Proof. Let $\mathbf{y}_1$ and $\mathbf{y}_2$ be two solutions.

$$\begin{aligned}
&(c_1\mathbf{y}_1 + c_2\mathbf{y}_2)' && \\
=&c_1\mathbf{y}_1' + c_2\mathbf{y}_2' && \text{derivation is linear} \\
=&c_1A\mathbf{y}_1 + c_2A\mathbf{y}_2 && \mathbf{y}_1 \text{ og } \mathbf{y}_2 \text{ are solutions} \\
=&A(c_1\mathbf{y}_1 + c_2\mathbf{y}_2) && \text{Matrix multiplication is linear}
\end{aligned}$$

□

Example 8.0.1

Consider the system

$$\begin{array}{rcrcr}
y_1'(t) & = & y_1(t) & + & 2y_2(t) \\
y_2'(t) & = & 2y_1(t) & - & 2y_2(t)
\end{array}$$

which can be written as

$$\begin{bmatrix} y_1'(t) \\ y_2'(t) \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix}$$

Two solutions of this system (verify yourself) is

$$\begin{bmatrix} 2e^{2t} \\ e^{2t} \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} e^{2t}$$

and

$$\begin{bmatrix} e^{-3t} \\ -2e^{-3t} \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix} e^{-3t}$$

Initial value problem

As in the one-dimensional case there is a plethora of possible solutions to a system of ODEs. If you have a solution you can scale it arbitrary to get an infinite amount of other solutions. However, we will see that if we impose an extra condition we reduce the amount of solutions from the infinite to the singular. Such a condition is as always called an initial condition,

$$\mathbf{y}(0) = \mathbf{b}, \qquad \mathbf{b} \in \mathbb{R}^n$$

Definition 8.0.1

An initial value problem in systems of ODEs is a system

$$\mathbf{y}' = A\mathbf{y}$$

with a condition

$$\mathbf{y}(0) = \mathbf{b}$$

In the one-dimensional case we saw that $y' = ay$ $y(0) = y_0$ was solved by

$$y = y_0 e^{at}$$

and we will in fact see that our Initial value problem here will have the solution

$$\mathbf{y} = e^{At}\mathbf{b}$$

To understand this we first need to find out what the **matrix exponential**,

$$e^{At}$$

is exactly. [2] The Taylor series of the ordinary exponential e^t is

$$e^t = 1 + t + \frac{1}{2}t^2 + \frac{1}{6}t^3 + \frac{1}{24}t^4 + \cdots + \frac{1}{k!}t^k + \cdots = \sum_{n=0}^{\infty} \frac{t^n}{n!}.$$

So we naively define the matrix exponential for a $n \times n$-matrix A by substituting t in the series above with At,

$$e^A = I_n + At + \frac{1}{2}(At)^2 + \frac{1}{6}(At)^3 + \cdots + \frac{1}{k!}(At)^k + \cdots = \sum_{n=0}^{\infty} \frac{1}{n!}(At)^n, \qquad \text{where } (At)^0 = I_n.$$

We will inherit some properties from the ordinary exponential, firstly we have

[2] In this note we will only scratch the surface when it comes to matrix exponentials. The interested or skeptical reader may want to consult other sources, for example the book on ODEs listed on the webpage.

Theorem 8.0.2

Let A be a $n \times n$-matrix, then

$$\frac{d}{dt}e^{At} = Ae^{At} = e^{At}A$$

In addition

Theorem 8.0.3

The exponential of the zero matrix is the identity matrix,

$$e^{0_n} = I_n$$

also, for every square matrix A, the matrix exponential e^{At} is invertible and the inverse is given by

$$(e^{At})^{-1} = e^{-At}$$

However, in general the property of $e^{a+b} = e^a e^b$ doesn't generalize except for commuting matrices, that is:

Theorem 8.0.4

If $AB = BA$, then

$$Ae^B = e^B A,$$
$$e^A e^b = e^b e^A$$

and

$$e^{A+B} = e^A e^B$$

Let us now state our main result which we revealed right above.

Theorem 8.0.5 Existence and uniqueness of solution

For every $\mathbf{b} \in \mathbb{R}^n$, the initial value problem

$$\mathbf{y}' = Ay, \qquad \mathbf{y}(0) = \mathbf{b}$$

has the unique solution

$$\mathbf{y} = e^{At}\mathbf{b}$$

8.1 Calculating the matrix exponential

If we have an invertible matrix P and a matrix B such that A can be decomposed as $A = PBP^{-1}$, then we have

$$\begin{aligned} e^{At} =& e^{PBP^{-1}t} \\ =& \sum_{n=0}^{\infty} \frac{1}{n!}(PBP^{-1}t)^n \\ =& \sum_{n=0}^{\infty} \frac{1}{n!}(P(Bt)P^{-1})^n \\ =& \sum_{n=0}^{\infty} \frac{1}{n!}P(B^n t)P^{-1} \\ =& P\left(\sum_{n=0}^{\infty} \frac{1}{n!}B^n t\right)P^{-1} \\ =& Pe^{Bt}P^{-1}. \end{aligned}$$

also, if $B = D$ is a diagonal matrix, i.e. A is diagonalizable, then we can explicitly find e^{Dt} as

$$e^{Dt} = \begin{bmatrix} e^{d_1 t} & & \text{\huge 0} \\ & \ddots & \\ \text{\huge 0} & & e^{d_n t} \end{bmatrix}.$$

Diagonalizable

Example 8.1.1

a)Find the matrix exponential e^{At} of

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

We start by finding the eigenvalues and eigenvectors of the matrix. The characteristic polynomial of A is

$$\begin{aligned} \rho_A(\lambda) = \begin{vmatrix} 1-\lambda & 2 \\ 2 & 1-\lambda \end{vmatrix} = (1-\lambda)^2 - 4 =& \lambda^2 - 2\lambda - 3 \\ =& (\lambda-3)(\lambda+1) \end{aligned}$$

so the eigenvalues of the matrix are $\lambda_1 = 3$ and $\lambda_2 = -1$. Let us now find the eigenvectors belonging to λ_1:

$$\mathrm{Null}(A - 3I_2) = \mathrm{Null}\begin{bmatrix} -2 & 2 \\ 2 & -2 \end{bmatrix} = \mathrm{Null}\begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}$$

So an eigenvector of λ_1 is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. Now for the eigenvectors of λ_2:

$$\mathrm{Null}(A + 1I_2) = \mathrm{Null}\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} = \mathrm{Null}\begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right\}$$

So an eigenvector of λ_2 is $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

A has two linearly independent eigenvectors and is therefore diagonalizable with

$$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad P = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

The inverse matrix P^{-1} is given by

$$P^{-1} = \frac{1}{-2}\begin{bmatrix} -1 & -1 \\ -1 & 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

Since we now have that $A = PDP^{-1}$, we can calculate the exponentials using what we learnt above

$$\begin{aligned} e^{At} &= e^{PDtP^{-1}} = Pe^{Dt}P^{-1} \\ &= \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{3t} & 0 \\ 0 & e^{-t} \end{bmatrix}\frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} e^{3t} & e^{3t} \\ e^{-t} & -e^{-t} \end{bmatrix} \\ &= \frac{1}{2}\begin{bmatrix} e^{3t} + e^{-t} & e^{3t} - e^{-t} \\ e^{3t} - e^{-t} & e^{3t} + e^{-t} \end{bmatrix} \end{aligned}$$

b) Solve the initial value problem

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}\mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

We know that the solution of initial value problem is given by

$$\mathbf{y} = e^{At}\begin{bmatrix}2\\0\end{bmatrix}$$

and since we have already calculated e^{At} we have

$$\begin{aligned}\mathbf{y} &= \frac{1}{2}\begin{bmatrix}e^{3t}+e^{-t} & e^{3t}-e^{-t}\\ e^{3t}-e^{-t} & e^{3t}+e^{-t}\end{bmatrix}\begin{bmatrix}2\\0\end{bmatrix}\\ &= \begin{bmatrix}e^{3t}+e^{-t}\\ e^{3t}-e^{-t}\end{bmatrix}\\ &= \begin{bmatrix}1\\1\end{bmatrix}e^{3t} + \begin{bmatrix}1\\-1\end{bmatrix}e^{-t}\end{aligned}$$

General solution when diagonalizable

Now, let us try to find a solution of

$$\mathbf{y}' = A\mathbf{y}, \qquad \mathbf{y}(0) = \mathbf{b}$$

when $A = PDP^{-1}$ is diagonalizable. Then we can do a substitution, by introducing the new vector $\mathbf{c} = P^{-1}\mathbf{b}$. The matrix P consists of linearly independent eigenvectors $\mathbf{v}_1, \dots, \mathbf{v}_n$ belonging to the eigenvalues $\lambda_1, \dots, \lambda_n$ in corresponding entries of the diagonal in D, so we have that the solution is

$$\begin{aligned}\mathbf{y} &= e^{At}\mathbf{b} = Pe^{Dt}P^{-1}\mathbf{b} = Pe^{Dt}\mathbf{c}\\ &= \begin{bmatrix}\mathbf{v}_1 & \cdots & \mathbf{v}_n\end{bmatrix}\begin{bmatrix}e^{\lambda_1} & & \Huge 0\\ & \ddots & \\ \Huge 0 & & e^{\lambda_n}\end{bmatrix}\begin{bmatrix}c_1\\ \vdots c_n\end{bmatrix}\\ &= c_1e^{\lambda_1 t}\mathbf{v}_1 + c_2e^{\lambda_2 t}\mathbf{v}_2 + \cdots c_ne^{\lambda_n t}\mathbf{v}_n\end{aligned}$$

Let us summarize this in a result

Theorem 8.1.1 Diagonalizable matrices

Let A be a real valued $n \times n$ matrix with n linearly independent eigenvectors, then the general solution of

$$\mathbf{y}' = A\mathbf{y}$$

is on the form

$$c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t} + \cdots + c_n\mathbf{v}_ne^{\lambda_n t}$$

where

$$\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$$

are the linearly independent eigenvectors of A,

$$\lambda_1, \lambda_2, \ldots, \lambda_n$$

are the corresponding eigenvalues (possibly with repetition of values), and

$$c_1, c_2, \ldots, c_n$$

are scalars.

Example 8.1.2

Consider the system $\mathbf{y}' = A\mathbf{y}$ with

$$\begin{bmatrix} 1 & 2 & 0 & 0 \\ 0 & 2 & 0 & 3 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 0 & 4 \end{bmatrix}.$$

The matrix is upper triangular so we read out the eigenvalues from the diagonal as 1, 2 og 4. Eigenvectors of these are respectively (check yourself)

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \text{ og } \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix}$$

The general solution of the system is therefore

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} e^t + c_2 \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} e^{2t} + c_3 \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} e^{2t} + c_4 \begin{bmatrix} 2 \\ 3 \\ 3 \\ 2 \end{bmatrix} e^{4t}.$$

Jordan form

All $n \times n$-matrices A can be written on the form PJP^{-1} for some invertible matrix P and a matrix in what is called Jordan form J. When the matrix in question is diagonalizable, then J is diagonal and we can do as above. In other cases, J is nearly diagonal. Let us look at the case when J is on the form

$$J = \begin{bmatrix} \lambda & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & \lambda & 1 & 0 & \cdots & 0 & 0 & 0 \\ & & \lambda & 1 & \cdots & 0 & 0 & 0 \\ & & & \lambda & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & \lambda & 1 & 0 \\ & & & & & & \lambda & 1 \\ & & & & & & & \lambda \end{bmatrix}$$

In particular we now know that A has the eigenvalue λ as its only eigenvalue. We can choose to write J as a sum of matrices $\lambda I_n + N$ where N is

$$N = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & & 0 & 1 & \cdots & 0 & 0 & 0 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 1 & 0 \\ & & & & & & 0 & 1 \\ & & & & & & & 0 \end{bmatrix}$$

Now, we observe that $e^{Jt} = e^{\lambda I_n t} e^{Nt}$, since $(\lambda I_n)N = N(\lambda I_n)$. Also, $e^{\lambda t I_n}$ is equal to $e^{\lambda t} I_n$ since λI_n is diagonal. Further, we can see that $N^n = 0$,

$$N^2 = \begin{bmatrix} 0 & 0 & 1 & 0 & \cdots & 0 & 0 & 0 \\ & 0 & 0 & 1 & \cdots & 0 & 0 & 0 \\ & & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & 1 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

$$N^{n-3} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 1 & 0 & 0 \\ & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ & & 0 & 0 & \cdots & 0 & 0 & 1 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}, \quad N^{n-2} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 1 & 0 \\ & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ & & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}, \quad N^{n-1} = \begin{bmatrix} 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 1 \\ & 0 & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & 0 & 0 & \cdots & 0 & 0 & 0 \\ & & & 0 & \cdots & 0 & 0 & 0 \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 0 & 0 & 0 \\ & & & & & & 0 & 0 \\ & & & & & & & 0 \end{bmatrix}$$

so the series expression of e^{Nt} will terminate after n terms,

$$e^{Nt} = I + tN + \frac{1}{2}(tN)^2 + \cdots + \frac{1}{(n-1)!}(tN)^{n-1} + \mathbf{0}$$

this gives us

$$e^{Jt} = e^{\lambda t} \begin{bmatrix} 1 & t & t^2/2 & t^3/6 & \cdots & t^{n-3}/(n-3)! & t^{n-2}/(n-2)! & t^{n-1}/(n-1)! \\ & 1 & t & t^2/2 & \cdots & t^{n-4}/(n-4)! & t^{n-3}/(n-3)! & t^{n-2}/(n-2)! \\ & & 1 & t & \cdots & t^{n-5}/(n-5)! & t^{n-4}/(n-4)! & t^{n-3}/(n-3)! \\ & & & 1 & \cdots & t^{n-6}/(n-6)! & t^{n-5}/(n-5)! & t^{n-4}/(n-4)! \\ & & & & \ddots & \vdots & \vdots & \vdots \\ & & & & & 1 & t & t^2/2 \\ & & & & & & 1 & t \\ & & & & & & & 1 \end{bmatrix}$$

Let us consider a small example.

Example 8.1.3

a) Find e^{At} when

$$A = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}.$$

$$A = -1I_2 + \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$$

so we have

$$e^{At} = e^{-t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

b) Solve the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \qquad \mathbf{y}(0) = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We know that the solution is given by

$$\mathbf{y} = e^{At}\mathbf{y}(0) = e^{-t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = e^{-t} \begin{bmatrix} -1+t \\ 1 \end{bmatrix}$$

Example 8.1.4

Let A be the matrix

$$\begin{bmatrix} -7 & 9 \\ -4 & 5 \end{bmatrix}.$$

The characteristic polynomial is $\rho_A(\lambda) = (\lambda+1)^2$, so A has the eigenvalue $\lambda = -1$

with algebraic multiplicity 2. The eigenspace is spanned by $\mathbf{v} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$. We can't find another linearly independent eigenvector of A, but we can find a **generalized eigenvector** $\mathbf{w} = \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}$ which satisfies

$$(A - \lambda I)\mathbf{w} = \mathbf{v}.$$

The matrix $P = \begin{bmatrix} \mathbf{v} & \mathbf{w} \end{bmatrix} = \begin{bmatrix} 3 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}$ with columns $\mathbf{v}$ and $\mathbf{w}$ is invertable with inverse

$$P^{-1} = \begin{bmatrix} 0 & \frac{1}{2} \\ -2 & 3 \end{bmatrix}.$$

From this we have

$$A = PJP^{-1}$$

where J is the matrix $J = \begin{bmatrix} -1 & 1 \\ 0 & -1 \end{bmatrix}$ as in the last example.

The matrix exponential e^{At} is therefore given as

$$e^{At} = Pe^{Jt}P^{-1}$$

Now, if we would like to have the general solution of the system $\mathbf{y}' = A\mathbf{y}$, then we can do as for the diagonalizable case above, namely a variable change. Set $\mathbf{x}(t) = P{-}1\mathbf{y}(t)$, then we have that the general solution of

$$\mathbf{x}' = J\mathbf{x}$$

is equal to

$$\mathbf{x}(t) = e^{Jt}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} t \\ 1 \end{bmatrix}$$

The general solution to $\mathbf{y}' = A\mathbf{y}$ is therefore

$$\begin{aligned} \mathbf{y}(t) = P\mathbf{x}(t) &= \begin{bmatrix} 3 & -\frac{1}{2} \\ 2 & 0 \end{bmatrix}\left(c_1e^{-t}\begin{bmatrix} 1 \\ 0 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} t \\ 1 \end{bmatrix}\right) \\ &= c_1e^{-t}\begin{bmatrix} 3 \\ 3 \end{bmatrix} + c_2e^{-t}\begin{bmatrix} 3t - 1/2 \\ 2t \end{bmatrix} \\ &= c_1e^{-t}\begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2e^{-t}\left(t\begin{bmatrix} 3 \\ 2 \end{bmatrix} + \begin{bmatrix} -\frac{1}{2} \\ 0 \end{bmatrix}\right). \end{aligned}$$

From this example we obtain a procedure for finding the general solution of

2×2-systems with only one eigenvalue.

Theorem 8.1.2

Let A be a real 2×2-matrix with a real eigenvalue λ of algebraic multiplicity 2. Let $\mathbf{v}$ be an eigenvector of λ, and let $\mathbf{w}$ be a vector solving $(A - \lambda I)\mathbf{w} = \mathbf{v}$. Then the solutions of the system $\mathbf{y}' = A\mathbf{y}$ are on the form

$$\mathbf{y}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\lambda t}(t\mathbf{v} + \mathbf{w}).$$

Complex eigenvalues

If A is a 2×2-matrix with complex eigenvalues $a \pm ib$ and real coefficients, we already know that the general solution of $\mathbf{y}' = A\mathbf{y}$ is given by

$$\mathbf{y}(t) = c_1 e^{a+ib}\mathbf{v}_1 + c_2 e^{a-ib}\mathbf{v}_2$$

where $\mathbf{v}_1$ and $\mathbf{v}_2$ are eigenvectors belonging to the eigenvalues. However, these solutions are in general complex-valued. Hence, we can ask ourselves whether we can find a general expression for the real-valued solutions.

We start by stating a result without proving or arguing for it.

Theorem 8.1.3

Let A be a real 2×2-matrix with a complex eigenvalue $\lambda = a - bi$, $b \neq 0$, and let $\mathbf{v} = \begin{bmatrix} \alpha_1 + i\beta_1 \\ \alpha_2 + i\beta_2 \end{bmatrix} \in \mathbb{C}^2$ be an eigenvector belonging to λ. Then we can factor A as following:

$$A = PCP^{-1} \text{ with } P = \begin{bmatrix} \mathrm{Re}\mathbf{v} & \mathrm{Im}\mathbf{v} \end{bmatrix} = \begin{bmatrix} \alpha_1 & \beta_1 \\ \alpha_2 & \beta_2 \end{bmatrix}$$

and

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}.$$

The form $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ is particularly nice, since we can rewrite it as a sum of commuting matrices

$$C = aI_2 + bS, \quad \text{for } S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The commutativity of them gives us that

$$e^{Ct} = e^{aI}e^{bSt} = e^{at}e^{bSt}$$

Let us try to find the last matrix exponential. We first observe that S^i is periodic

$$S^0 = I_2, \quad S^1 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad S^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = -I_2, \quad S^3 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = -S$$

$$S^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2, \quad S^5 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} = S, \quad S^6 = -I_2 = S^2, \quad S^7 = -S = S^3, \quad \ldots$$

In the power series of e^{bSt} we can group together the even and odd powers of S, and get

$$\begin{aligned} e^{bSt} =& \left(1 - \frac{1}{2!}(bt)^2 + \frac{1}{4!}(bt)^4 - \frac{1}{6!}(bt)^6 + \frac{1}{8!}(bt)^8 - \cdots\right) I_2 \\ &+ \left(bt - \frac{1}{3!}(bt)^3 + \frac{1}{5!} - \frac{1}{7!}(bt)^7 + \frac{1}{9!}(bt)^9 - \cdots\right) S \end{aligned}$$

Recognize that the first part is equal to the taylor series of $\cos(bt)$

$$\cos(bt) = 1 - \frac{1}{2!}(bt)^2 + \frac{1}{4!}(bt)^4 - \frac{1}{6!}(bt)^6 + \frac{1}{8!}(bt)^8 - \cdots = \sum_{n=0}^{\infty}(-1)^n \frac{x^{2n}}{(2n)!}$$

and the second part is equal to the taylor series of $\sin(bt)$

$$\sin(bt) = bt - \frac{1}{3!}(bt)^3 + \frac{1}{5!} - \frac{1}{7!}(bt)^7 + \frac{1}{9!}(bt)^9 - \cdots = \sum_{n=0}^{\infty}(-1)^{n-1} \frac{x^{2n+1}}{(2n+1)!}.$$

Hence, we see that

$$e^{bSt} = \cos(bt)I_2 + \sin(bt)S = \begin{bmatrix} \cos(bt) & -\sin(bt) \\ \sin(bt) & \cos(bt) \end{bmatrix}$$

Now, using the substitution $\mathbf{x} = P^{-1}\mathbf{y}$ in $\mathbf{y}' = A\mathbf{y}$, with P as in the above theorem, we get

$$\mathbf{x}(t) = c_1 e^{at} \begin{bmatrix} \cos(bt) \\ \sin bt \end{bmatrix} + c_2 e^{at} \begin{bmatrix} -\sin(bt) \\ \cos(bt) \end{bmatrix}$$

and using $\mathbf{y}(t) = P\mathbf{x}(t)$

$$\begin{aligned} \mathbf{y}(t) =& \begin{bmatrix} \mathrm{Re}\mathbf{v} & \mathrm{Im}\mathbf{v} \end{bmatrix} \left(c_1 e^{at} \begin{bmatrix} \cos(bt) \\ \sin bt \end{bmatrix} + c_2 e^{at} \begin{bmatrix} -\sin(bt) \\ \cos(bt) \end{bmatrix} \right) \\ =& c_1 e^{at}\cos(bt)\mathrm{Re}(\mathbf{v}) + c_1 e^a t \sin(bt)\mathrm{Im}(\mathbf{v}) \\ &- c_2 e^{at}\sin(bt)\mathrm{Re}(\mathbf{v}) + c_2 e^{at}\cos(bt)\mathrm{Im}(\mathbf{v}) \\ =& c_1 e^{at}[\cos(bt)\mathrm{Re}(\mathbf{v}) + \sin(bt)\mathrm{Im}(\mathbf{v})] \\ &+ c_2 e^{at}[\cos(bt)\mathrm{Im}(\mathbf{v}) - \sin(bt)\mathrm{Re}(\mathbf{v})] \end{aligned}$$

Theorem 8.1.4

Let A be a 2×2-matrix with a complex eigenvalue $a - ib$ and real entries. Let also $\mathbf{v}$ be an eigenvector of $a - ib$, then the general solution of the system

$$\mathbf{y}' = A\mathbf{y}$$

is given by

$$\begin{aligned}\mathbf{y}(t) =& c_1 e^{at}[\cos(bt)\mathrm{Re}(\mathbf{v}) + \sin(bt)\mathrm{Im}(\mathbf{v})] \\ &+ c_2 e^{at}[\cos(bt)\mathrm{Im}(\mathbf{v}) - \sin(bt)\mathrm{Re}(\mathbf{v})]\end{aligned}$$

Example 8.1.5

Let us find the solutions of the system $\mathbf{y}' = A\mathbf{y}$ with

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$

The eigenvalues are $\pm i$. You can verify that $\mathbf{v} = \begin{bmatrix} i \\ 1 \end{bmatrix}$ is an eigenvector belonging to i. We find

$$\mathrm{Re}\mathbf{v} = \mathrm{Re}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{Re} i \\ \mathrm{Re} 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

and

$$\mathrm{Im}\mathbf{v} = \mathrm{Im}\begin{bmatrix} i \\ 1 \end{bmatrix} = \begin{bmatrix} \mathrm{Im} i \\ \mathrm{Im} 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Theorem 8.1.4 tells us that the general solution is spanned by

$$\begin{aligned}& e^{\alpha t}(\mathrm{Re}(\mathbf{v})\cos(\beta t) - \mathrm{Im}(\mathbf{v})\sin(\beta t)) \\ =& e^{0 \cdot t}(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\cos(1 \cdot t) - \begin{bmatrix} 1 \\ 0 \end{bmatrix}\sin(1 \cdot t)) \\ =& \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix}\end{aligned}$$

and

$$\begin{aligned} & e^{\alpha t}(\mathrm{Re}(\mathbf{v})\sin(\beta t) + \mathrm{Im}(\mathbf{v})\cos(\beta t)) \\ =& e^{0\cdot t}(\begin{bmatrix}0\\1\end{bmatrix}\sin(1\cdot t) + \begin{bmatrix}1\\0\end{bmatrix}\cos(1\cdot t)) \\ =& \begin{bmatrix}\cos t\\ \sin t\end{bmatrix}. \end{aligned}$$

So the general solution is

$$\begin{aligned} & c_1\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} + c_2\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix} \\ =& \begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}. \end{aligned}$$

We recognize this as

$$\mathbf{y}(t) = e^{At}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

Example 8.1.6

Let

$$A = \begin{bmatrix}1 & -1\\ 1 & 1\end{bmatrix}.$$

The complex eigenvalues are $1 \pm i$. Verify that $\begin{bmatrix}i\\ 1\end{bmatrix}$ is an eigenvector. Theorem 8.1.4 tell us that the general solution is spanned by

$$e^t\begin{bmatrix}\cos t\\ \sin t\end{bmatrix} \quad \text{and} \quad e^t\begin{bmatrix}-\sin t\\ \cos t\end{bmatrix}.$$

The general solution is

$$e^t\begin{bmatrix}\cos t & -\sin t\\ \sin t & \cos t\end{bmatrix}\begin{bmatrix}c_1\\ c_2\end{bmatrix}.$$

which we recognize as

$$\mathbf{y}(t) = e^{At}\begin{bmatrix}c_1\\ c_2\end{bmatrix}$$

8.2 From nth order to system

The observant student may already have seen that the three cases above seem quite similar to the solution cases for a second order ODE. This is not just a quirky coincidence, but a consequence of the following relations.

Say that we have a third order linear ODE with constant coefficients

$$y''' + a_2 y'' + a_1 y' + a_0 y = 0$$

In order to solve this we could try to introduce a few more functions, say

$$\begin{aligned} x_1(t) &= y(t) \\ x_2(t) &= y'(t) \\ x_3(t) &= y''(t) \end{aligned}$$

then we can rewrite our ODE as a system of first order ODEs

$$\begin{array}{rcrrr} x_1' & = & & x_2 & \\ x_2' & = & & & x_3 \\ x_3' & = & -a_0 x_1 & -a_1 x_2 & -a_2 x_3 \end{array}$$

or in matrix notation

$$\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -a_0 & -a_1 & -a_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

In general we have

Theorem 8.2.1 Converting to system

An nth-order ODE

$$y^{(n)} + a_{n-1} y^{(n-1)} + \cdots + a_2 y'' + a_1 y' + a_0 y = f(t)$$

can be converted to a system of first order ODEs

$$\begin{bmatrix} y_1' \\ y_2' \\ y_3' \\ \vdots \\ y_{n-1}' \\ y_n' \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 1 \\ -a_0 & -a_1 & -a_2 & -a_3 & \cdots & -a_{n-1} \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \\ \vdots \\ y_{n-1} \\ y_n \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 0 \\ f(t) \end{bmatrix}$$

where

$$
\begin{aligned}
y_1 &= y \\
y_2 &= y' \\
y_3 &= y'' \\
&\vdots \\
y_{n-1} &= y^{(n-2)} \\
y_n &= y^{(n-1)}
\end{aligned}
$$

Second order ODEs

Now, let us consider the homogeneous second order ODEs,

$$y''(t) + a_1 y'(t) + a_0 y(t) = 0$$

Rewriting this as a system, we get

$$\mathbf{y}' = \begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \mathbf{y}$$

We have seen how we can solve these systems above, we start by finding the characteristic polynomial

$$\rho(\lambda) = \begin{vmatrix} -\lambda & 1 \\ -a_0 & -a_1 - \lambda \end{vmatrix} = \lambda(\lambda + a_1) - (-a_0) = \lambda^2 + \lambda a_1 + a_0$$

and then depending on whether this polynomial has real, complex or a single root we find eigenvector(s) (and generalized eigenvectors). We can now see that this polynomial is the same that we considered when solving the second order ODEs earlier. Let us venture on further, we claim that if λ is a root of $\rho(\lambda)$ (i.e. an eigenvalue), then $\begin{bmatrix} 1 \\ \lambda \end{bmatrix}$ is an eigenvector of the matrix:

$$
\begin{aligned}
\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ \lambda \end{bmatrix} &= \begin{bmatrix} \lambda \\ -a_0 - a_1\lambda \end{bmatrix} \\
&= \begin{bmatrix} \lambda \\ \lambda^2 \end{bmatrix} \text{ since } \lambda \text{ is a root of } \rho(\lambda) \\
&= \lambda \begin{bmatrix} 1 \\ \lambda \end{bmatrix}.
\end{aligned}
$$

Two real eigenvalues

Now, if

$$\begin{bmatrix} 0 & 1 \\ -a_0 & -a_1 \end{bmatrix}$$

has two real eigenvalues, we know that the system is solved by

$$\mathbf{y} = c_1 e^{\lambda t} \begin{bmatrix} 1 \\ \lambda_1 \end{bmatrix} + c_2 e^{\lambda t} \begin{bmatrix} 1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t} \\ c_1 \lambda_1 e^{\lambda_1 t} + c_2 \lambda_2 e^{\lambda_2 t} \end{bmatrix}$$

and if we read out the first entry of the solutions we rediscover the solution $y(t)$ of the ODE:

$$y(t) = y_1(t) = c_1 e^{\lambda_1 t} + c_2 e^{\lambda_2 t}$$

Only one eigenvalue

The only way for us to get exactly one eigenvalue is if the expression under the root sign in

$$\lambda = \frac{-a_1 \pm \sqrt{a_1^2 - 4a_0}}{2}$$

is zero. That is $a_1^2 = 4a_0$ and $\lambda = \frac{-a_1}{2}$.

$$A = \begin{bmatrix} 0 & 1 \\ -a_0 & -4a_0 \end{bmatrix}$$

In order to find a solution of the system we therefore need to find a generalized eigenvector, that is solve

$$\left(\begin{bmatrix} 0 & 1 \\ -a_0 & -4a_0 \end{bmatrix} - \begin{bmatrix} -\frac{a_0}{2} & 0 \\ 0 & -\frac{a_0}{2} \end{bmatrix} \right) \mathbf{w} = \begin{bmatrix} 1 \\ \lambda \end{bmatrix}$$

The left hand side is equal to

$$\begin{bmatrix} \frac{a_1}{2} & 1 \\ -a_0 & -\frac{a_0}{2} \end{bmatrix} \mathbf{w}$$

and we see that $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is a solution. Thus our general solution of the system is

$$\begin{aligned} \mathbf{y}(t) &= c_1 e^{\lambda t} \mathbf{v} + c_2 e^{\lambda t} (t\mathbf{v} + \mathbf{w}) \\ &= c_1 e^{-a_0/2t} \begin{bmatrix} 1 \\ -a_0/2 \end{bmatrix} + c_2 e^{-a_0/2} \left(t \begin{bmatrix} 1 \\ -a_0/2 \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \\ &= \begin{bmatrix} c_1 e^{-a_0/2t} + c_2 t e^{-a_0/2t} \\ c_1 e^{a_0 t}(1 - a_0/2) - c_2 t e^{-a_0 t} \end{bmatrix} \end{aligned}$$

The solution of the ODE is now read out from the first entry,

$$y(t) = y_1(t) = c_1 e^{-a_0/2t} + c_2 t e^{-a_0/2t} = c_1 e^{\lambda t} + c_2 t e^{\lambda t}$$

which is the same solution as we had earlier.

Two complex eigenvalues

If we have comples eigenvalues $\lambda_{\pm} = a \pm ib$, then from Theorem 8.1.4 the solution of the system is given by

$$\begin{aligned}\mathbf{y}(t) =& c_1 e^{at}[\cos(bt)\mathrm{Re}(\mathbf{v}) + \sin(bt)\mathrm{Im}(\mathbf{v})] \\ &+ c_2 e^{at}[\cos(bt)\mathrm{Im}(\mathbf{v}) - \sin(bt)\mathrm{Re}(\mathbf{v})]\end{aligned}$$

Further, since we know that an eigenvector is given by

$$\mathbf{v} = \begin{bmatrix} 1 \\ \lambda_- \end{bmatrix} = \begin{bmatrix} 1 \\ a - ib \end{bmatrix},$$

we have

$$\mathrm{Re}\mathbf{v} = \begin{bmatrix} 1 \\ a \end{bmatrix} \quad \text{and} \quad \mathrm{Im}\mathbf{v} = \begin{bmatrix} 0 \\ b \end{bmatrix}$$

and

$$\begin{aligned}\mathbf{y}(t) =& c_1 e^{at}\left[\cos(bt)\begin{bmatrix} 1 \\ a \end{bmatrix} + \sin(bt)\begin{bmatrix} 0 \\ b \end{bmatrix}\right] \\ &+ c_2 e^{at}\left[\cos(bt)\begin{bmatrix} 0 \\ b \end{bmatrix} - \sin(bt)\begin{bmatrix} 1 \\ a \end{bmatrix}\right]\end{aligned}$$

If we read out the first entry of this solution we get the solution of the ODE

$$y(t) = y_1(t) = e^{at}[c_1\cos(bt) + c_2 \sin bt]$$

which also is the one we obtained earlier.

8.3 Phase portrait

As when we worked with ODEs in the start, we can obtain some information about the solutions of a system of ODEs through a graphical representation. Specifically the systems consisting of two ODEs.

We know that the system

$$\mathbf{y}' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\mathbf{y}$$

has a unique solution for every initial condition $\mathbf{y}(0)$. These solutions can be drawn as trajectories in the y_1y_2-plane.

Example 8.3.1

Consider the system

$$\mathbf{y}' = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} \mathbf{y}$$

which has general solution

$$\mathbf{y}(t) = c_1 e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

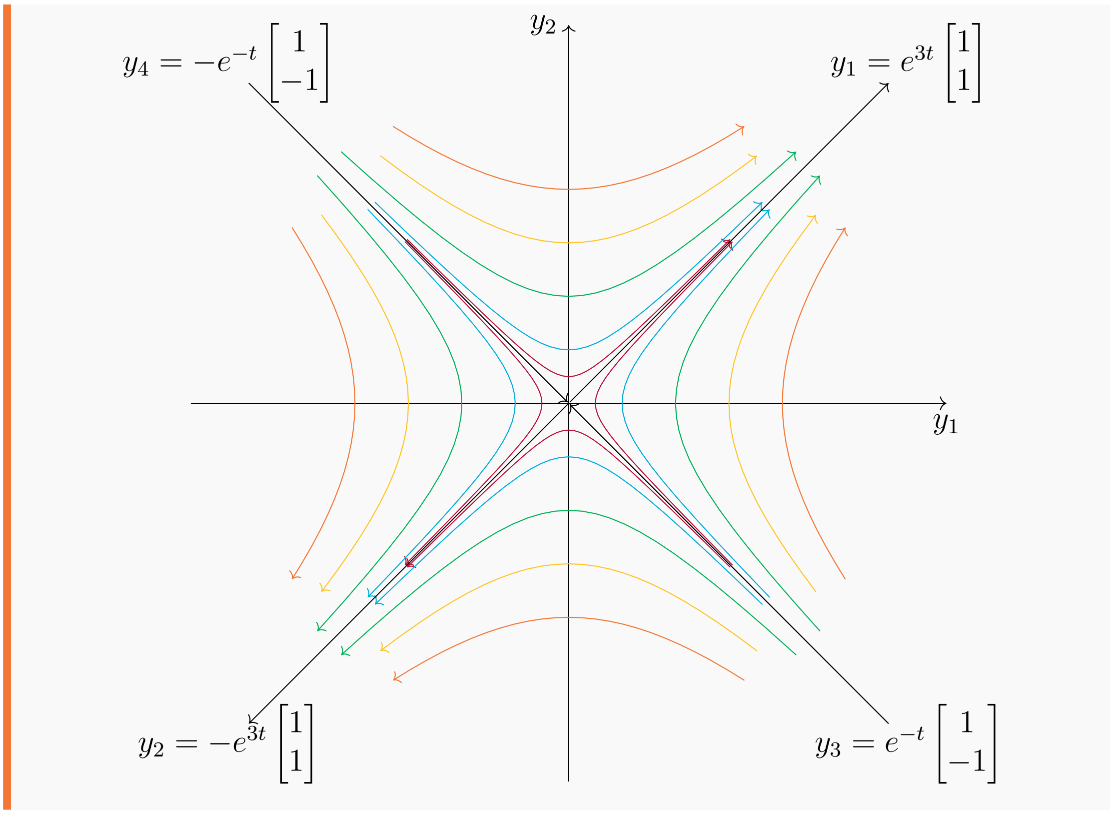
Below we have drawn different solutions of this system. Note specifically the black solution curves that correspond to

$$y_1(t) = e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_2(t) = -e^{3t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$y_3(t) = e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$y_4(t) = -e^{-t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

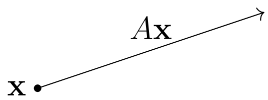


Vector field

To each 2×2 system

$$\mathbf{y}' = A\mathbf{y}$$

we have an associated vector field. For each point $\mathbf{x}$ in the $\mathbf{y}_1\mathbf{y}_2$-plane we can associate a vector $A\mathbf{x}$ starting in $\mathbf{x}$.

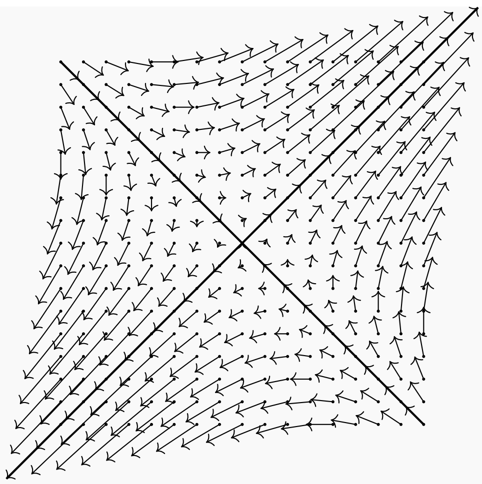


$A\mathbf{x}$ as a vector starting in $\mathbf{x}$

Example 8.3.2

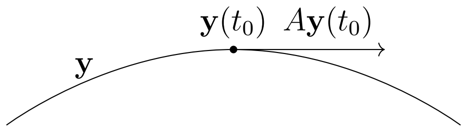
Here is a sketch of the vector field associated to the system with

$$A = \begin{bmatrix}1 & 2\\2 & 1\end{bmatrix}.$$



The axes in bold are the lines spanned by the eigenvectors $\begin{bmatrix}1\\1\end{bmatrix}$ og $\begin{bmatrix}1\\-1\end{bmatrix}$ which correspond to the eigenvectors 3 og -1. Do notice how the arrows move towards infinity along $\begin{bmatrix}1\\1\end{bmatrix}$ which correspond to a positive eigenvalue; towards the origin along $\begin{bmatrix}1\\-1\end{bmatrix}$ which correspond to a negative eigenvalue.

How does the vector field help us to understand the solution of systems? A solution $\mathbf{y}$ of $\mathbf{y}' = A\mathbf{y}$ is a curve which satisfies the derivative in a point of time t_0, $\mathbf{y}'(t_0) = A\mathbf{y}(t_0)$. The derivative is in other words, the vector in $\mathbf{y}(t_0)$ from the vector field associated to A.



The vector $A\mathbf{y}(t_0)$ is the derivative of $\mathbf{y}$ in $t = t_0$

The arrows lie tangent to the solution curves.

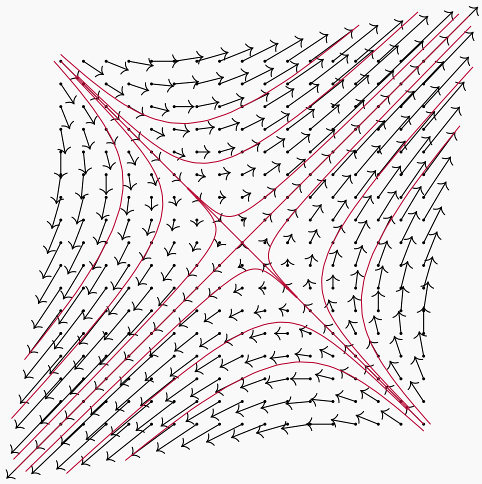
A phase portrait of $\mathbf{y}' = A\mathbf{y}$ is a sketch of all possible solutions, including the the orientation, i.e. which direction we move along the curves for increasing t. One way of making a phase portrait is to first make a sketch of the vector field of A and then drawing curves along the arrows.

Example 8.3.3

Let us look back at the system with

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}.$$

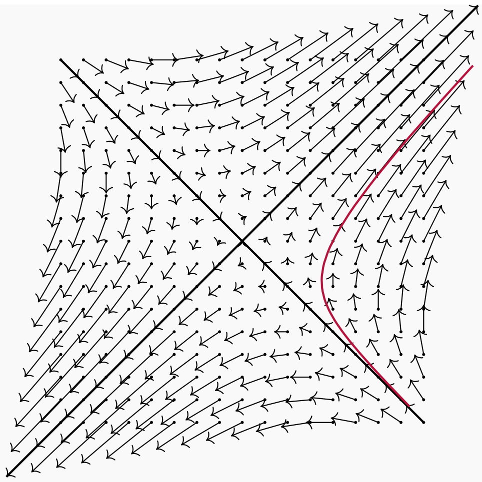
Draw some curves that have the arrows in the vector field of A as tangents to obtain a phase portrait:



The solutions move towards the origin along $\operatorname{span}\left(\begin{bmatrix} 1 \\ -1 \end{bmatrix}\right)$, and then bends before moving away from the origin along $\operatorname{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right)$. The reason for this behavior is that the term $c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-t}$ dominates for negative t (negative eigenvalue), and $c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{3t}$ dominates for positive t (positive eigenvalue).

Example 8.3.4

When we look at a solution that moves through a given point $\mathbf{y}(0) = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$, we have a unique curve in the vector field:



Let us now discuss in more detail two of the different cases of solutions we have for a two dimensional system, namely

1. two different real eigenvalues, and
2. two complex eigenvalues.

Two real eigenvalues

Let $\mathbf{v}_1$ and $\mathbf{v}_2$ be two linearly independent eigenvectors belonging to the eigenvalues λ_1 og λ_2. Then we know that

$$c_1\mathbf{v}_1e^{\lambda_1 t} + c_2\mathbf{v}_2e^{\lambda_2 t}$$

is a general solution of $\mathbf{y}' = A\mathbf{y}$.

We can classify all possible phase portraits based on the eigenvalues and eigenvectors. The factor $e^{\lambda t}$ tells us how the solutions moves along the span of $\mathbf{v}$ when t changes:

λ	$e^{\lambda t}$	$\mathbf{v}e^{\lambda t}$
> 0	increases	away from the origin
$= 0$	constant	don't move
< 0	decreases	towards the origin

Table 8.1: What happens when t grows?

Be aware that $\lambda < 0$ dominates when $t \ll 0$; $\lambda > 0$ dominates when $t \gg 0$. For a given system we have two such terms in the solution Here is a method to sketch the phase portrait of $\mathbf{y}' = A\mathbf{y}$ when A have two real eigenvalues:

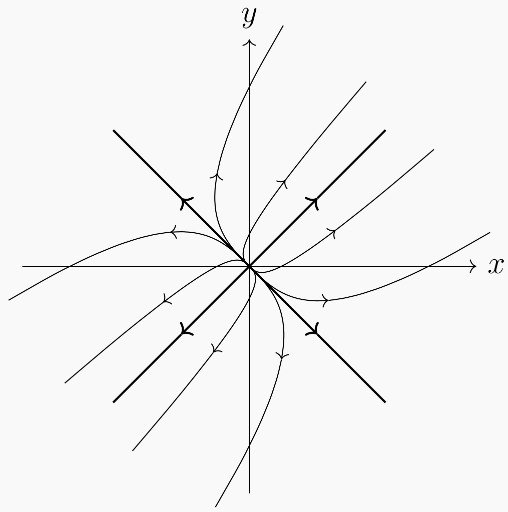
(a) Draw the span of two linearly independent eigenvectors in the plane.
(b) Determine the movement of the solutions along the span of each eigenvector.
(c) Draw curves that move according to point b)

Example 8.3.5

The system with

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

has two linearly independent eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ with eigenvalues 3 and 1. From the discussion above, we know that the solutions move away from the origin along both eigenvectors. Notice that $e^{3t}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ dominates for big t. The solutions between the axis spanned by the eigenvectors will therefore become increasingly parallell to $\pm\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ when t grows larger.



Two complex eigenvalues

When we have complex eigenvalues we know that the solution is given by

$$\begin{aligned} \mathbf{y}(t) =& c_1 e^{at}[\cos(bt)\mathrm{Re}(\mathbf{v}) + \sin(bt)\mathrm{Im}(\mathbf{v})] \\ &+ c_2 e^{at}[\cos(bt)\mathrm{Im}(\mathbf{v}) - \sin(bt)\mathrm{Re}(\mathbf{v})] \end{aligned}$$

The terms with cosine and sine gives a circular movement. If $\alpha \neq 0$ we also have in addition an inward or outward motion, dependent on the sign of α –as in the real case. The combination of these two motion are spirals which either moves inward to the origin or outward from the origin. Vi oppsummerer:

α	movement
> 0	outward going spirals
$= 0$	circular
< 0	inward moving spirals

Example 8.3.6

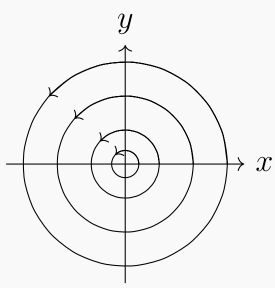
Let

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

be as in the example (8.1.5), where we found the general solution

$$\begin{aligned} & c_1 \begin{bmatrix} \cos t \\ \sin t \end{bmatrix} + c_2 \begin{bmatrix} -\sin t \\ \cos t \end{bmatrix} \\ = & \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}. \end{aligned}$$

The phase portrait consists of circles centered in the origin, oriented anti-clockwise.



Example 8.3.7

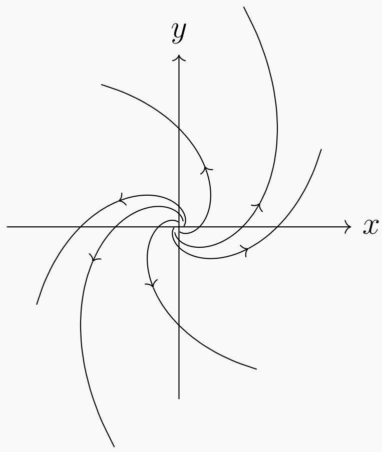
Let

$$A = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

be as in Example (8.1.6). We found the solution

$$e^t \begin{bmatrix} \cos t & -\sin t \\ \sin t & \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

The factor e^t contributes to an outward motion, while the matrix gives a circular motion in an anti-clockwise direction. The phase portrait consists of outward moving spirals oriented anti-clockwise.



We knew that the motion was anti-clockwise since we recognized the rotation matrix. A more methodical way would be to plot the vectorfield of A in a point or two. Often $\begin{bmatrix}1\\0\end{bmatrix}$ or $\begin{bmatrix}0\\1\end{bmatrix}$ are convenient points for these, since they are easily plotted. For example

$$A\begin{bmatrix}1\\0\end{bmatrix}=\begin{bmatrix}1\\1\end{bmatrix}$$

so the vector field is sloped upwards in the first quadrant from $\begin{bmatrix}1\\0\end{bmatrix}$. Thus the spirals, which are tangent to the field, moves anti-clockwise.

Problem Set - Systems of ODEs

1. Find e^{At} when

a)

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

ab

$$A = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}$$

d)

$$A = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}$$

2. Find the solutions of the initial value problems

a)

$$\mathbf{y}' = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$$

b)

$$\mathbf{y}' = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

c)

$$A = \begin{bmatrix} 2 & 1 \\ 0 & 2 \end{bmatrix}, \quad \mathbf{y}(0) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$$

d)

$$\mathbf{y}' = \frac{1}{2}\begin{bmatrix} \sqrt{2} & -\sqrt{2} \\ \sqrt{2} & \sqrt{2} \end{bmatrix}\mathbf{y}, \quad \mathbf{y}(0) = \begin{bmatrix} 4 \\ 4 \end{bmatrix}$$

3. Find e^{At} when

a)

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 0 & 3 & 5 \\ 0 & 0 & 5 \end{bmatrix}$$

4. Find the general solution of $A\mathbf{y} = \mathbf{y}'$, when

a)

$$A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

b)

$$A = \begin{bmatrix} -6 & -11 & 16 \\ 2 & 5 & -4 \\ -4 & -5 & 10 \end{bmatrix}$$

Solutions - Systems of ODEs

1. a We recall from yesterday that

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

is diagonalizable as

$$A = PDP^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Thus

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} e^0 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

1. b We recall from yesterday that

$$A = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

is diagonalizable as

$$A = PDP^{-1} = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix}.$$

Thus

$$\begin{aligned} e^{At} = Pe^{Dt}P^{-1} &= \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} -1/2 & 1/2 \\ 1/2 & 1/2 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & e^t \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \frac{1}{2} \begin{bmatrix} 1 + e^t & -1 + e^t \\ -1 + e^t & 1 + e^t \end{bmatrix} \end{aligned}$$

1. c We recognize A as being a Jordan block, so we now that

$$A = 2I_2 + N = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} + \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

And since $2I_2$ and N commute ($2I_2N = 2N = N2 = NI_22 = N(2I_2)$), we have

$$e^{At} = e^{2I_2t}e^{Nt}.$$

Now,

$$e^{2I_2t} = \begin{bmatrix} e^{2t} & 0 \\ 0 & e^{2t} \end{bmatrix} = e^{2t}I_2$$

and

$$\begin{aligned} e^{Nt} &= I_2 + \frac{t}{1!}N + \frac{t^2}{2!}N^2 + \frac{t^3}{3!}N^3 + \cdots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{t}{1!}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} + \frac{t^2}{2!}\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \cdots = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \frac{t}{1!}\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \end{aligned}$$

gives us

$$e^{At} = e^{2t}\begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix}$$

1. d We start by finding the eigenvalues of A.

$$\begin{aligned} \rho_A(\lambda) = \begin{vmatrix} \frac{1}{\sqrt{2}} - \lambda & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} - \lambda \end{vmatrix} &= (\frac{1}{\sqrt{2} - \lambda})^2 + \frac{1}{2} \\ &= \frac{1}{2} - \frac{2}{\sqrt{2}} + \lambda^2 + \frac{1}{2} \\ &= \lambda^2 - \sqrt{2}\lambda + 1 \end{aligned}$$

$$\lambda = \frac{\sqrt{2} \pm \sqrt{2-4}}{2} = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$$

We have complex eigenvalues, so we try to factor A as PCP^{-1}, where

$$P = \begin{bmatrix} \mathrm{Re}\mathbf{v} & \mathrm{Im}\mathbf{v} \end{bmatrix}$$

for any eigenvector $\mathbf{v}$ of A, and

$$C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$$

where $\lambda = a \pm ib = \frac{1}{\sqrt{2}} \pm i\frac{1}{\sqrt{2}}$. The observant reader may already have figured out that $A = C$, and we could skip past finding the eigenvectors. For the not so observant reader we head on with seemingly non-interesting calculations for the eigenvector.

The eigenspace of $\lambda = \frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}$ is found as

$$E(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}) = \mathrm{Null}\left(\begin{bmatrix} i\frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & i\frac{1}{\sqrt{2}} \end{bmatrix}\right) = \mathrm{Null}\left(\begin{bmatrix} i & -1 \\ 0 & 0 \end{bmatrix}\right) = \mathrm{span}\left(\begin{bmatrix} 1 \\ i \end{bmatrix}\right)$$

We therefore have that $\begin{bmatrix} 1 \\ i \end{bmatrix}$ is an eigenvector, and since $\mathrm{Re}\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathrm{Im}\mathbf{v} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, we see that

$$P = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

and

$$A = PCP^{-1} = I_2 C I_2 = C$$

which the observant reader saw a few unnecessary calculations ago.

Now, we rewrite A as

$$A = \frac{1}{\sqrt{2}} I_2 + \frac{1}{\sqrt{2}} S, \text{ for } S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

and get

$$e^{At} = e^{t/\sqrt{2}} e^{1/\sqrt{2}\ S} = e^{t/\sqrt{2}} \begin{bmatrix} \cos(t/\sqrt{2}) & -\sin(t/\sqrt{2}) \\ \sin(t/\sqrt{2}) & \cos(t/\sqrt{2}) \end{bmatrix}$$

2. We know that the solution of the initial value problem

$$\mathbf{y}' = A\mathbf{y}, \quad \mathbf{y}(0) = \mathbf{b}$$

is given by

$$\mathbf{y} = e^{At}\mathbf{b}.$$

From the first problem we therefore read out the solutions:

a)

$$\mathbf{y} = e^{At} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} e^t & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 3e^t \\ 2 \end{bmatrix}$$

b)

$$\mathbf{y} = e^{At} \begin{bmatrix} 1 \\ 1 \end{bmatrix} = e^t \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

c)

$$\mathbf{y} = e^{At} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = e^{2t} \begin{bmatrix} 1 & t \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = e^{2t} \begin{bmatrix} 1+4t \\ 4 \end{bmatrix}$$

d)

$$\mathbf{y} = e^{At} \begin{bmatrix} 4 \\ 4 \end{bmatrix} = 4e^{t/\sqrt{2}} \begin{bmatrix} \cos(t/\sqrt{2}) - \sin(t/\sqrt{2}) \\ \cos(t/\sqrt{2}) + \sin(t/\sqrt{2}) \end{bmatrix}$$

3. a In the lecture yesterday, we found that

$$A = \begin{bmatrix} 1 & 3 \\ 4 & -3 \end{bmatrix}$$

can be diagonalized as

$$A = PDP^{-1} = \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \frac{1}{-8} \begin{bmatrix} -2 & -1 \\ -2 & 3 \end{bmatrix} = \frac{1}{8} \begin{bmatrix} 3 & 1 \\ 2 & -2 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & -5 \end{bmatrix} \begin{bmatrix} 2 & 1 \\ 2 & -3 \end{bmatrix}.$$

Thus

$$\begin{aligned} e^{At} &= \frac{1}{8}\begin{bmatrix}3 & 1\\ 2 & -2\end{bmatrix}\begin{bmatrix}e^{3t} & 0\\ 0 & e^{-5t}\end{bmatrix}\begin{bmatrix}2 & 1\\ 2 & -3\end{bmatrix}\\ &= \frac{1}{8}\begin{bmatrix}3e^{3t} & e^{-5t}\\ 2e^{3t} & -2e^{-5t}\end{bmatrix}\begin{bmatrix}2 & 1\\ 2 & -3\end{bmatrix}\\ &= \frac{1}{8}\begin{bmatrix}6e^{3t}+2e^{-5t} & 3e^{3t}-3e^{-5t}\\ 4e^{3t}-4e^{-5t} & 2e^{3t}+6e^{-5t}\end{bmatrix}\end{aligned}$$

3. b In yesterdays exercises we found the diagonalization

$$\begin{aligned} A &= PDP^{-1}\\ &= \begin{bmatrix}1 & 3 & 25\\ 0 & 1 & 15\\ 0 & 0 & 6\end{bmatrix}\begin{bmatrix}2 & 0 & 0\\ 0 & 3 & 0\\ 0 & 0 & 5\end{bmatrix}\begin{bmatrix}1 & -3 & 10/3\\ 0 & 1 & -5/2\\ 0 & 0 & 1/6\end{bmatrix}\end{aligned}$$

Thus we have

$$e^{At} = \begin{bmatrix}1 & 3 & 25\\ 0 & 1 & 15\\ 0 & 0 & 6\end{bmatrix}\begin{bmatrix}e^{2t} & 0 & 0\\ 0 & e^{3t} & 0\\ 0 & 0 & e^{5t}\end{bmatrix}\begin{bmatrix}1 & -3 & 10/3\\ 0 & 1 & -5/2\\ 0 & 0 & 1/6\end{bmatrix}$$

We could be satisfied with this expression and go on with our day, or we could insist on calculating the product. You might want to let a computer do it for you, but in any case you should get

$$e^{At} = \begin{bmatrix}e^{2t} & 3e^{3t}-3e^{2t} & e^{2t}\cdot\ 10/3+3e^{3t}\left(-5/2\right)+25e^{5t}\cdot\ 1/6\\ 0 & e^{3t} & e^{3t}\left(-5/2\right)+15e^{5t}\cdot\ 1/6\\ 0 & 0 & 6e^{5t}\cdot\ 1/6\end{bmatrix}.$$

4. a We found in yesterdays exercises that A has eigenvalues $\lambda_1 = -1$ and $\lambda_2 = 2$. The eigenspace of λ_1 is spanned by

$$\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix} \text{ and } \begin{bmatrix}-1\\ 0\\ 1\end{bmatrix},$$

and the eigenspace of λ_2 is spanned by

$$\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}.$$

The general solution of $A\mathbf{y} = \mathbf{y}'$ is therefore given by

$$\mathbf{y} = c_1e^{-t}\begin{bmatrix}-1\\ 1\\ 0\end{bmatrix} + c_2e^{-t}\begin{bmatrix}-1\\ 0\\ 1\end{bmatrix} + c_3e^{2t}\begin{bmatrix}1\\ 1\\ 1\end{bmatrix}.$$

4. b The eigenvalues of A is $\lambda_1 = 2$, $\lambda_2 = 3$ and $\lambda_3 = 4$, and we can find corresponding eigenvectors as

$$\mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}, \quad \text{and } \mathbf{v}_3 = \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}.$$

A general solution is therefore

$$\mathbf{y} = c_1 e^{2t} \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{3t} \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix} + c_3 e^{4t} \begin{bmatrix} 7 \\ -2 \\ 3 \end{bmatrix}.$$

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