Norwegian University of Science and Technology
Department of Mathematical Sciences

Faglig kontakt under eksamen:
Kari Hag tlf. 73 59 35 21
Ivar Amdal tlf. 73 59 34 68

EKSAMEN I TMA4115 MATEMATIKK 3

Engelsk
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Kl. 9–14

Hjelpemidler (kode C): Enkel kalkulator (HP30S), med tilhørende bruksanvisning
Rottman: *Matematisk formelsamling*

Sensurdato: 23. juni

Give reasons for all answers (with the exception of Problem 4).

Problem 1 *Complex numbers*

a) Write the complex number

$$w = \left(-\frac{\sqrt{3}}{4} + \frac{i}{4}\right)^3$$

in the form $re^{i\theta}$. Find all complex numbers z such that $z^3 = w$. Write the answer in the form $a + ib$ where a and b are real numbers. Use exact values for a and b.

b) Let z be an arbitrary complex number. A square $OABC$ in the complex plane has one corner in the origin O, and the corners are given counter clockwise. If the corner A is the number z, what are the corners B and C expressed by z?

Problem 2 *First order differential equations*

a) Solve the initial value problem

$$y' + \frac{2}{x}y = \frac{\cos x}{x^2}, \qquad y(\pi/2) = 0.$$

b) Use Euler's method with step size $h = 0.5$ to find approximates $y_1 \approx y(2.5)$ and $y_2 \approx y(3.0)$ to the solution $y(x)$ of the initial value problem

$$y' = 1 + (x-y)^2, \qquad y(2) = 1.$$

Problem 3 *Second order differential equations*

a) Find a general solution of the differential equation

$$y'' + y' - 2y = e^x + e^{2x}.$$

b) The differential equation

$$(*) \qquad (1-x^2)y'' + 2xy' - 2y = 0, \qquad -1 < x < 1$$

has two solutions of the form $y_1 = x + a$ and $y_2 = x^2 + b$ where a and b are constants. Find a and b by substitution. Explain why the solutions y_1 and y_2 are linearly independent, and give general solution of $(*)$.

c) Find a particular solution of the differential equations

$$(1-x^2)y'' + 2xy' - 2y = 6\left(1-x^2\right)^2, \qquad -1 < x < 1.$$

Problem 4 *Multiple choice problem - to be answered without explanations.*

a) Let A and B be $2 \times 2-$ matrices. If the determinant of A is 2 and the determinant of B is 3, what is the determinant of $C = -2A^{-1}B^T$?

A: 3 **B:** −3 **C:** 6 **D:** −6

b) For which (n) c are the vectors $\mathbf{v}_1 = (1,3,-3)$, $\mathbf{v}_2 = (-2,4,1)$, $\mathbf{v}_3 = (-1,1,c)$ linearly independent?

A: No c **B:** $c = 1$ **C:** $c \neq 1$ **D:** All c

Problem 5 *Matrices and systems of linear equations*
Given the matrix

$$A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 2 & -2 & 3 \\ 2 & 4 & -3 & 5 \end{bmatrix}.$$

a) Solve the system of equations $A\mathbf{x} = \mathbf{0}$.

b) Find a basis for the row space $\mathrm{Row}(A)$ and for the column space $\mathrm{Col}(A)$.

c) Show that the vector $\mathbf{y} = (1, 5, -3)$ lies in $\mathrm{Col}(A)^{\perp}$. What other vectors does $\mathrm{Col}(A)^{\perp}$ consist of? Give reason for your answer.

Problem 6 *Eigenvalues and eigenvectors*

a) Find the eigenvalues of the matrix

$$A = \begin{bmatrix} 1 & 1 & -2 \\ 4 & 0 & 4 \\ 1 & -1 & 4 \end{bmatrix}.$$

Show that $\mathbf{v}_1 = (-1, 3, 1)$ and $\mathbf{v}_2 = (0, 2, 1)$ are eigenvalues of A by computing $A\mathbf{v}_1$ and $A\mathbf{v}_2$.

b) Find an eigenvector $\mathbf{v}_3$ of A such that $\mathbf{v}_1$, $\mathbf{v}_2$ and $\mathbf{v}_3$ are linearly independent. Write up an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

c) Let $y_1 = y_1(t)$, $y_2 = y_2(t)$ and $y_3 = y_3(t)$ be differentiable functions of t. Solve the system of differential equations

$$\begin{aligned} y_1' &= y_1 + y_2 - 2y_3 \\ y_2' &= 4y_1 \qquad\;\; + 4y_3 \\ y_3' &= y_1 - y_2 + 4y_3 \end{aligned}$$

with initial conditions $y_1(0) = 0$, $y_2(0) = 1$ and $y_3(0) = 2$.

Problem 7 *Symmetric matrices*
Show in general that if *all* eigenvalues of a symmetric $n \times n$ – matrix A are positive $(\lambda > 0)$, then $\mathbf{x}^T A\mathbf{x} > 0$ for all vectors $\mathbf{x} \neq \mathbf{0}$ in $\mathbb{R}^n$.