Problem 1  Complex numbers

a) Write the complex number

\[ w = \left( -\frac{\sqrt{3}}{4} + \frac{i}{4} \right)^3 \]

in the form \( re^{i\theta} \). Find all complex numbers \( z \) such that \( z^3 = w \). Write the answer in the form \( a + ib \) where \( a \) and \( b \) are real numbers. Use exact values for \( a \) and \( b \).

b) Let \( z \) be an arbitrary complex number. A square \( OABC \) in the complex plane has one corner in the origin \( O \), and the corners are given counter clockwise. If the corner \( A \) is the number \( z \), what are the corners \( B \) and \( C \) expressed by \( z \)?
Problem 2  First order differential equations
a) Solve the initial value problem
\[ y' + \frac{2}{x} y = \frac{\cos x}{x^2}, \quad y(\pi/2) = 0. \]
b) Use Euler's method with step size \( h = 0.5 \) to find approximates \( y_1 \approx y(2.5) \) and \( y_2 \approx y(3.0) \) to the solution \( y(x) \) of the initial value problem
\[ y' = 1 + (x - y)^2, \quad y(2) = 1. \]

Problem 3  Second order differential equations
a) Find a general solution of the differential equation
\[ y'' + y' - 2y = e^x + e^{2x}. \]
b) The differential equation
\[ (1 - x^2)y'' + 2xy' - 2y = 0, \quad -1 < x < 1 \]
(*)
has two solutions of the form \( y_1 = x + a \) and \( y_2 = x^2 + b \) where \( a \) and \( b \) are constants. Find \( a \) and \( b \) by substitution. Explain why the solutions \( y_1 \) and \( y_2 \) are linearly independent, and give general solution of (*)
c) Find a particular solution of the differential equations
\[ (1 - x^2)y'' + 2xy' - 2y = 6(1 - x^2)^2, \quad -1 < x < 1. \]

Problem 4  Multiple choice problem - to be answered without explanations.
a) Let \( A \) and \( B \) be \( 2 \times 2 \) - matrices. If the determinant of \( A \) is 2 and the determinant of \( B \) is 3, what is the determinant of \( C = -2A^{-1}B^T \)?
A: 3  B: -3  C: 6  D: -6
b) For which \( n \) \( c \) are the vectors \( v_1 = (1, 3, -3) \), \( v_2 = (-2, 4, 1) \), \( v_3 = (-1, 1, c) \) linearly independent?
A: No \( c \)  B: \( c = 1 \)  C: \( c \neq 1 \)  D: All \( c \)
Problem 5  \textit{Matrices and systems of linear equations}

Given the matrix
\[ A = \begin{bmatrix} 1 & 2 & 1 & 0 \\ 1 & 2 & -2 & 3 \\ 2 & 4 & -3 & 5 \end{bmatrix}. \]

a) Solve the system of equations $Ax = 0$.

b) Find a basis for the row space $\text{Row}(A)$ and for the column space $\text{Col}(A)$.

c) Show that the vector $y = (1, 5, -3)$ lies in $\text{Col}(A)$. What other vectors does $\text{Col}(A)$ consist of? Give reason for your answer.

Problem 6  \textit{Eigenvalues and eigenvectors}

a) Find the eigenvalues of the matrix
\[ A = \begin{bmatrix} 1 & 1 & -2 \\ 4 & 0 & 4 \\ 1 & -1 & 4 \end{bmatrix}. \]

Show that $v_1 = (-1, 3, 1)$ and $v_2 = (0, 2, 1)$ are eigenvalues of $A$ by computing $Av_1$ and $Av_2$.

b) Find an eigenvector $v_3$ of $A$ such that $v_1$, $v_2$ and $v_3$ are linearly independent. Write up an invertible matrix $P$ and a diagonal matrix $D$ such that $P^{-1}AP = D$.

c) Let $y_1 = y_1(t)$, $y_2 = y_2(t)$ and $y_3 = y_3(t)$ be differentiable functions of $t$. Solve the system of differential equations
\begin{align*}
y'_1 &= y_1 + y_2 - 2y_3 \\
y'_2 &= 4y_1 + 4y_3 \\
y'_3 &= y_1 - y_2 + 4y_3
\end{align*}

with initial conditions $y_1(0) = 0$, $y_2(0) = 1$ and $y_3(0) = 2$.

Problem 7  \textit{Symmetric matrices}

Show in general that if all eigenvalues of a symmetric $n \times n$ matrix $A$ are positive ($\lambda > 0$), then $x^T Ax > 0$ for all vectors $x \neq 0$ in $\mathbb{R}^n$. 