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Contact during the exam:
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EXAM IN COURSE TMA4110 Calculus 3

English
Tuesday November 30, 2004
Hours: 9-13
Aids: C. (Approved calculator, Rottmann: *Matematisk formelsamling, English)*
Grading finished: December 21, 2004

It should be clearly stated how all answers are obtained.

Problem 1

Find all complex numbers z such that

$$z^3 = 1 + \sqrt{3}\,i.$$

Write the solutions in polar form, $re^{i\theta}$. Sketch the solutions in the complex plane.

Problem 2

(a) Solve the initial value problem $y'' + 9y = 0$, $y(0) = 1$, $y'(0) = 6$.

(b) Find the general solution to the equation $y'' + 9y = 6e^{3x} + \sin 3x$.

(c) Find the general solution to the equation $x^2y'' + 2xy' - 6y = 0, x > 0$.

Problem 3

Let

$$A = \begin{bmatrix} 2 & 1 & 2 & 1 & 4 \\ 1 & 1 & 0 & 1 & 1 \\ -1 & 0 & -2 & 0 & -3 \\ 0 & -1 & 2 & -1 & 5 \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 4 \end{bmatrix}.$$

(a) Solve the system of equations $A\mathbf{x} = \mathbf{b}$ by bringing the total matrix (extended coefficient matrix) over to reduced Echelon form.

(b) Find a basis for $\mathrm{Row}\,(A)$, $\mathrm{Col}\,(A)$, and $\mathrm{Row}\,(A)^{\perp}$. State the dimension of each of these vector spaces.

Problem 4

Let A be the matrix

$$A = \begin{bmatrix} -2 & -2 & 4 \\ -4 & 0 & 4 \\ -4 & -2 & 6 \end{bmatrix}.$$

(a) Show that the eigenvalues of A are $\lambda = 0$ and $\lambda = 2$. Find a basis for each eigenspace.

(b) If possible, find an invertible matrix P and a diagonal matrix D such that $P^{-1}AP = D$.

(c) Solve the following system of differential equations

$$\begin{aligned} y_1' &= -2y_1 - 2y_2 + 4y_3, \\ y_2' &= -4y_1 + 4y_3, \\ y_3' &= -4y_1 - 2y_2 + 6y_3, \end{aligned}$$

when $y_1(0) = 0,\ y_2(0) = 3,\ y_3(0) = 1$.

Problem 5

Lake A in Bymarka is to be treated with rothenon poison by pouring M kilos of rothenon into the lake at $t = 0$. A river is flowing from A into lake B. Lake A has a water volume V and B a volume $3V$. The amount of water (per time unit, m^3/s) out from A is U, whereas the amount of water out from B is $6U$. We assume that the mixture in lake A and lake B is uniform and that the volumes of A and B are constant.

(a) Show that the amounts of rothenon in A, $y_1(t)$, and B, $y_2(t)$, satisfy the following system of differential equations:

$$\frac{dy_1(t)}{dt} = -U\frac{y_1(t)}{V},$$
$$\frac{dy_2(t)}{dt} = U\frac{y_1(t)}{V} - 2U\frac{y_2(t)}{V}, \quad t \geq 0.$$

What are the initial conditions for y_1 and y_2?

(b) Find the solution of the system in **(a)** when $U/V = 1$ and $M = 1$.

Problem 6

In a 3×3-matrix the sum of the elements in each row is equal to 4. Show that such a matrix always has an eigenvector $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. What is the corresponding eigenvalue?