EXAM IN TMA4110 CALCULUS 3
English
Wednesday December 20, 2006
Time: 9–13

You may use the following (code C): Approved calculator (HP30S)
Rottman: Matematisk formelsamling

Grades to be announced: January 19, 2007

All answers have to be justified (with the exception of Problem 3). When grading, the 12
problems (1, 2abc, 3, 4, 5abc, 6ab, 7) will, as a rule, have the same weight.

Problem 1  Solve the equation

\[ z^3 = \frac{4}{1 - i\sqrt{3}}. \]

Write the solutions in the form \( re^{i\theta} \). Use a figure to show the location of the solutions in the
complex plane.

Problem 2  Find a general solution of the differential equations
a) \( y'' - y' + 2.5y = 0, \)
b) \( y'' + y' - 2y = x + e^{-2x}, \)
c) \( 4x^2y'' + 8xy' - 3y = 0, \quad x > 0. \)
Problem 3  Multiple choice problem — to be answered without explanations by choosing one alternative for each question.

Let \( y_1(x) \) and \( y_2(x) \) be two solutions of the constant coefficient equation \( y'' + ay' + by = 0 \). If the Wronskian \( W(y_1, y_2) \) equals 0 when \( x = 0 \), what is the value of \( W(y_1, y_2) \) when \( x = 1 \)?

A: 0  B: 1  C: \( e \)  D: The value depends on \( a \) and \( b \).

Which one of the vectors \( v_1, v_2, v_3, v_4 \) is in the null space \( \text{Null}(A) \) of the matrix

\[
A = \begin{bmatrix}
3 & 2 & 0 \\
0 & 1 & -3 \\
2 & 1 & 1 \\
\end{bmatrix}
\]

A: \( v_1 = (2, 1, 1) \)  B: \( v_2 = (2, -3, 1) \)  C: \( v_3 = (-2, 3, 1) \)  D: \( v_4 = (-1, 2, 0) \)

Problem 4  Four one-way streets in a city intersect as shown in the figure to the right. The number of cars passing per hour is shown in the figure.

Write down a system of equations in the form \( Ax = b \) which \( x = (x_1, x_2, x_3, x_4) \) must satisfy, and solve the system.

Find \( x_1, x_2 \) and \( x_3 \) if the \( x_4 \)-section is closed for traffic, making \( x_4 = 0 \).

Problem 5  A matrix \( A \) and a vector \( b \) is given by

\[
A = \begin{bmatrix}
1 & 1 & -2 & 0 & -1 \\
2 & 2 & -1 & 0 & 1 \\
-1 & -1 & 2 & -3 & 1 \\
0 & 0 & 1 & 1 & 1 \\
\end{bmatrix}, \quad b = \begin{bmatrix}
b_1 \\
b_2 \\
b_3 \\
b_4 \\
\end{bmatrix}.
\]

a) Find a basis for the row space \( \text{Row}(A) \) and a basis for the column space \( \text{Col}(A) \).

b) What is the dimension of \( \text{Row}(A) \), \( \text{Col}(A) \) and the orthogonal complement \( \text{Row}(A)^	op \)?

c) Show that the orthogonal complement \( \text{Col}(A)^	op \) is spanned by the vector \( u = (3, -1, 1, 3) \). Find, for example by using \( u \), a condition which \( b_1, b_2, b_3 \) and \( b_4 \) must satisfy in order that the system \( Ax = b \) shall be consistent.
Problem 6

a) Compute the eigenvalues of the matrix

\[ A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}. \]

Find an orthogonal matrix \( P \) with determinant equal to 1 such that \( P^T A P \) is a diagonal matrix.

b) A conic section has equation

\[ 3x^2 + 4xy - 1 = 0. \]

Introduce a new, rotated coordinate system such that the conic section is in standard position relative to the new coordinate system. Make a figure to show the location of the conic section and the new coordinate axes in the \( xy \)-plane.

Problem 7

Let \( A \) be an \( n \times n \)-matrix.
Show that if 5 is an eigenvalue of \( A \), then 3 is an eigenvalue of \( A - 2I \) where \( I \) denotes the \( n \times n \) identity matrix.

Also show that if \( \mathbf{v} \) is an \( n \)-vector satisfying \( A\mathbf{v} \neq \mathbf{0} \) and \( A^2\mathbf{v} = \mathbf{0} \), then \( \mathbf{v} \) and \( A\mathbf{v} \) are linear independant.