EXAM IN TMA4115 CALCULUS 3
English
Wednesday, June 7, 2006
Time: 9–13

You may use the following (code C): Approved calculator (HP30S)
Rottman: Matematisk formelsamling

Grades to be announced: June 28.

All answers have to be justified (with the exception of Problem 4).

Problem 1 Calculate

\[ w = (\sqrt{3} + i)^4. \]

Find the solution of the equation \( z^4 = w \) lying in the second quadrant in the complex plane.

Problem 2

a) Solve the initial value problem

\[ y'' + 6y' + 10y = 0, \quad y(0) = 0, \quad y'(0) = 2. \]

b) Find a general solution of the differential equation

\[ y'' - 3y' + 2y = e^x + 10 \sin x. \]

c) Suppose that \( p \) and \( q \) are functions such that the differential equation

\[ y'' + p(x)y' + q(x)y = 0, \quad x > 0 \]

has a basis \( y_1 = x^2, y_2 = x^2 \ln x \) of solutions. Find a general solution of the equation

\[ y'' + p(x)y' + q(x)y = x, \quad x > 0. \]
Problem 3  
Given the matrix

\[ A = \begin{bmatrix}
1 & 2 & 1 & 2 & 1 \\
0 & 0 & 1 & -1 & -1 \\
2 & 4 & 3 & 3 & 1 \\
1 & 2 & 2 & 1 & 1
\end{bmatrix}. \]

a) Solve the system of equations \( Ax = 0 \), and find a basis for \( \text{Null}(A) \).

b) Find a basis for each of the spaces \( \text{Col}(A) \), \( \text{Row}(A) \) and \( \text{Row}(A)^\perp \).

c) Show that the vector \( v = (2, 1, -1, 0) \) is in the orthogonal complement \( \text{Col}(A)^\perp \). Is \{v\} a basis for \( \text{Col}(A)^\perp \)?

Problem 4  
\textit{Multiple choice problem — to be answered without explanations.}

a) Find the least squares solution \((\bar{x}, \bar{y})\) of the system

\[
\begin{align*}
x + 3y &= 5 \\
x - y &= 1 \\
x + y &= 0.
\end{align*}
\]

A: (0, 1)  B: (1/2, 3/2)  C: (1, 1)  D: (3/2, 1/2)

b) For which real numbers \( \alpha, \beta \) is \( P \) an orthogonal matrix with determinant equal 1?

\[ P = \frac{1}{2} \begin{bmatrix}
\alpha & -1 & -1 & \beta \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
\alpha & 1 & 1 & \beta
\end{bmatrix} \]

A: \( \alpha = 2, \beta = 0 \)  B: \( \alpha = 1, \beta = -1 \)  C: \( \alpha = -1, \beta = 1 \)  D: \( \alpha = 0, \beta = 2 \)

Problem 5  
Given the matrix

\[ A = \begin{bmatrix}
3 & -1 & -2 \\
2 & 0 & -2 \\
2 & -1 & -1
\end{bmatrix}. \]

a) Show that \( A \) has eigenvalues 0 and 1. Find all the eigenvectors of \( A \).
b) Find an invertible matrix $P$ and a diagonal matrix $D$ such that

$$P^{-1}AP = D.$$ 

Solve the system of differential equations

$$
\begin{align*}
y' &= 3y_1 - y_2 - 2y_3 \\
y' &= 2y_1 - 2y_3 \\
y' &= 2y_1 - y_2 - y_3
\end{align*}
$$

with initial conditions $y_1(0) = 0$, $y_2(0) = 1$ and $y_3(0) = 2$.

c) A square matrix $B$ is said to be idempotent if $B^2 = B$. Show that the matrix $A$ is idempotent. Show in general that if $\lambda$ is an eigenvalue of an idempotent matrix, then $\lambda = 0$ or $\lambda = 1$. 