EXAM IN TMA4110 CALCULUS 3
English
Monday, December 3, 2007
9 am – 1 pm

You may use the following (code C): Approved calculator (HP30S)
Rottman: Matematisk formelsamling

Grades to be announced: January 3, 2008

All answers have to be justified. When grading, the 12 problems (1, 2abc, 3, 4, 5, 6abc, 7ab) will, as a rule, have the same weight.

Problem 1
Write down the polar form of the complex number $iz(\bar{z})^{-1}$, where $z = re^{i\theta}$ and $z \neq 0$.
Show on a figure all solutions of the equation $iz = \bar{z}$.

Problem 2
a) The equation

$$y'' + ay' + by = 0$$

with constant coefficients has solutions $y_1 = e^{(3/2)x}$ and $y_2 = e^{-(1/2)x}$. Determine $a$ and $b$.

b) Solve the initial value problem

$$y'' - 4y' + 5y = 0, \quad y(0) = 1, \quad y'(0) = 4.$$ 

c) Find a general solution of the equation

$$y'' - 2y' - 15y = e^{5x} - 17 \cos 3x.$$
Problem 3  The equation
\[ y'' + 2(\tan x)y' - y = 0 \]
has a solution \( y_1(x) = \sin x \). Find another solution \( y_2(x) = u(x)y_1(x) \) where \( u \) is a non-constant function.

The following integral is given:
\[ \int \frac{\cos^2 x}{\sin^2 x} \, dx = -\frac{\cos x}{\sin x} - x + C. \]

Problem 4  We shall determine the temperatures \( T_1, T_2, T_3, T_4 \) at four points \( P_1, P_2, P_3, P_4 \) on the square plate, see the figure on the right. Temperatures along the sides of the plate are shown on the figure.

Set up and solve a system of equations for \( T_1, T_2, T_3, T_4 \) if we assume that the temperature at each point \( P_1 \) to \( P_4 \) is the average of the temperatures at the four neighboring points (to the right, above, to the left, and below).

Problem 5  Suppose that \( v_1, v_2, v_3 \) are linearly independent vectors in \( \mathbb{R}^n \). Are vectors
\[ w_1 = v_1 + v_2, \quad w_2 = v_1 + v_3, \quad w_3 = v_2 + v_3 \]
linearly dependent or linearly independent? (Remember to explain the answer.)

Problem 6
a) Find a basis for the solution space of the homogeneous system
\[ \begin{align*}
x_1 - 2x_2 + x_3 &= 0 \\
x_1 - x_3 + 2x_4 &= 0.
\end{align*} \]

b) Determine the orthogonal projection of the vector \((1, 2, -3, 1)\) onto the subspace \( V = \text{span}\{v_1, v_2\} \) of \( \mathbb{R}^4 \), where \( v_1 \) and \( v_2 \) are orthogonal vectors given by
\( v_1 = (1, -2, 1, 0), \quad v_2 = (1, 0, -1, 2). \)

c) Find also vectors \( v_3 \) and \( v_4 \) such that \( \{v_1, v_2, v_3, v_4\} \) is an orthogonal basis for \( \mathbb{R}^4 \).
Problem 7  The matrix $A$ is given by

$$A = \begin{bmatrix} 10 & -9 \\ 6 & -5 \end{bmatrix}.$$ 

a) Find the eigenvalues and eigenvectors of $A$, and write down a general solution of the system of differential equations

$$y'_1 = 10y_1 - 9y_2,$$
$$y'_2 = 6y_1 - 5y_2.$$ 

b) Determine an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$. Find a matrix $B$ such that $B^2 = A$. 