EXAM IN TMA4115 CALCULUS 3

English
Monday, May 21, 2007
9 am – 1 pm

You may use the following (code C): Simple calculator (HP30S) with user's manual
Rottman: Matematisk formelsamling

Grades to be announced: June 11, 2007

All answers have to be justified. When grading, the 12 problems (1, 2ab, 3ab, 4ab, 5ab, 6ab, 7) will, as a rule, have the same weight.

Problem 1
Show on a figure all complex numbers z such that

$$|z| = |z + 1|.$$  

Find the solutions of the equation

$$z^4 = (z + 1)^4.$$  

Problem 2

a) Solve the initial value problem

$$y'' + 2y' + y = 0, \quad y(0) = 2, \quad y'(0) = 3.$$  

b) Find a general solution of the differential equation

$$y'' + 2y' + y = x + 2e^{-x}.$$
Problem 3
a) Find a particular solution of the differential equation
\[ x^2 y'' - 4xy' + 4y = x^4, \quad x > 0. \]

b) Find a general solution of the differential equation
\[ (y'' + 4y)'' = y'' + 4y. \]

Problem 4
The matrix \( A = \begin{bmatrix} 1 & -1 & -2 & -2 & -4 \\ -2 & -4 & -2 & 1 & -1 \\ -2 & -1 & 1 & -2 & -1 \\ 0 & 1 & 1 & 0 & 1 \end{bmatrix} \) and the vector \( b = \begin{bmatrix} -2 \\ 1 \\ -2 \\ 0 \end{bmatrix} \) are given.

a) Solve the system \( Ax = b \).

b) Find a basis for \( \text{Null}(A) \), \( \text{Col}(A) \) and \( \text{Row}(A) \).

Problem 5
Let the vectors \((1,0,1,-1), (-1,1,0,1), (1,1,0,1)\) be a basis for a subspace \( V \) of \( \mathbb{R}^4 \).

a) Use the Gram-Schmidt algorithm to find an orthogonal basis for \( V \).

b) Find the orthogonal projection of \((1,1,3,4)\) into \( V \).

Problem 6
a) Show that the matrix \( A = \begin{bmatrix} -1 & -1 & 1 \\ -1 & -3 & 1 \\ -1 & -3 & 1 \end{bmatrix} \) has eigenvalues 0, -1, -2.

Find a matrix \( P \) and a diagonal matrix \( D \) such that \( A = PDP^{-1} \).

b) Find the solution \( x_1(t), x_2(t), x_3(t) \) of the system of differential equations
\[
\begin{align*}
x_1' &= -x_1 - x_2 + x_3 \\
x_2' &= -x_1 - 3x_2 + x_3 \\
x_3' &= -x_1 - 3x_2 + x_3
\end{align*}
\]
for which \( x_1(0) = 1, x_2(0) = -1, x_3(0) = 2 \).

Problem 7
Let \( A \) and \( B \) be \( m \times n \) matrices, and let \( C \) be a matrix such that \( A = BC \). Show that \( \text{Col}(A) \) is contained in \( \text{Col}(B) \). Also show that if \( C \) is invertible, then the rank of \( A \) equals the rank of \( B \).