Norwegian University of Science and Technology
Department of Mathematical Sciences

TMA4110 2016H – solutions

Problem 1

a) The characteristic equation, $\lambda^2 + 1 = 0$, has complex roots $\pm i = 0 \pm 1i$, so the general equation of the differential equation is $y(t) = e^{0t}(C_1 \cos 1t + C_2 \sin 1t) = C_1 \cos t + C_2 \sin t$.

b) To solve the differential equation, we need a particular solution. The general solution is then the sum of the particular solution and the general solution of the homogeneous equation above. We try the method of undetermined coefficients, and look for a particular solution of the form

$$y_{\mathrm{p}} = A \cos 2t + B \sin 2t + at^2 + bt + c.$$

Then

$$y_{\mathrm{p}}' = -2A \sin 2t + 2B \cos 2t + 2at + b \qquad \text{and} \qquad y_{\mathrm{p}}'' = -4A \cos 2t - 4B \sin 2t + 2a.$$

Substituting into the left-hand side of the equation, we get

$$\begin{aligned}(A \cos 2t + B \sin 2t + at^2 + bt + c) &+ (-4A \cos 2t - 4B \sin 2t + 2a)\\ &= -3A \cos 2t - 3B \sin 2t + at^2 + bt + 2a + c.\end{aligned}$$

This is equal to the right-hand side of the equation,

$$\sin 2t + t^2 + 1,$$

if $-3A = 0$, $-3B = 1$, $a = 1$, $b = 0$ and $2a + c = 1$, which is equivalent to $A = 0$, $B = -\frac{1}{3}$, $a = 1$, $b = 0$ and $c = -1$. Now we have a particular solution $y_{\mathrm{p}} = -\frac{1}{3} \sin 2t + t^2 - 1$, and the general solution is

$$y(t) = C_1 \cos t + C_2 \sin t - \frac{1}{3} \sin 2t + t^2 - 1.$$

Problem 2

a) To be a basis, $\mathcal{B}$ must span V and be linearly independent. By definition, V consists of all linear combinations of 1, x and x^2, so $\mathcal{B}$ spans V. To show that $\mathcal{B}$ is linearly independent, assume $c_0 \cdot 1 + c_1 x + c_2 x^2 = 0$ for all x (the right hand side is the zero polynomial, which is the zero vector of V). We must show that this is possible only when $c_0 = c_1 = c_2 = 0$. Inserting $x = 0, 1, -1$ in the equality, we get $c_0 = 0,\ c_0 + c_1 + c_2 = 0$ and $c_0 - c_1 + c_2 = 0$, and it is easy to see that this implies $c_0 = c_1 = c_2 = 0$, so that $\mathcal{B}$ is linearly independent, and thus a basis of V. (We could have used three other distinct values of x to show linear independence.)

T is a linear transformation if $T(f(x)+g(x)) = T(f(x))+T(g(x))$ for all $f(x)$, $g(x)$ in V and $T(cf(x)) = cT(f(x))$ for all $f(x)$ in V and c in $\mathbb{R}$. We have

$$\begin{aligned} T(f(x)+g(x)) &= (x+1)\tfrac{d}{dx}\big(f(x)+g(x)\big) + \big(f(x)+g(x)\big) \\ &= (x+1)\big(f'(x)+g'(x)\big) + f(x) + g(x) \\ &= \big((x+1)f'(x)+f(x)\big) + \big((x+1)g'(x)+g(x)\big) = T(f(x)) + T(g(x)) \end{aligned}$$

and

$$\begin{aligned} T(cf(x)) &= (x+1)\tfrac{d}{dx}\big(cf(x)\big) + cf(x) \\ &= (x+1)cf'(x) + cf(x) \\ &= c\big((x+1)f'(x) + f(x)\big) = cT(f(x)), \end{aligned}$$

showing that T is a linear transformation.

Let $f(x) = a_0 + a_1 x + a_2 x^2$. Then $T(f(x)) = (x+1)(a_1 + 2a_2 x) + a_0 + a_1 x + a_2 x^2 = (a_0 + a_1)\cdot 1 + (2a_1 + 2a_2)x + 3a_2 x^2$, so that

$$[f(x)]_{\mathcal{B}} = \begin{bmatrix} a_0 + a_1 \\ 2a_1 + 2a_2 \\ 3a_2 \end{bmatrix}.$$

On the other hand,

$$A[f(x)]_{\mathcal{B}} = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 2 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} a_0 + a_1 \\ 2a_1 + 2a_2 \\ 3a_2 \end{bmatrix},$$

so $[f(x)]_{\mathcal{B}} = A[f(x)]_{\mathcal{B}}$.

b) The 3×3 matrix A has 3 pivot columns, and is therefore invertible. The columns of an invertible matrix are independent and thus a basis of the column space, so the dimension of the column space is 3.

Problem 3

a) We find the characteristic polynomial,

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -\frac{1}{2} & -\frac{1}{8} \\ -\frac{1}{2} & \lambda & -\frac{7}{8} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = \begin{vmatrix} \lambda-1 & \lambda-1 & \lambda-1 \\ -\frac{1}{2} & \lambda & -\frac{7}{8} \\ -\frac{1}{2} & -\frac{1}{2} & \lambda \end{vmatrix} = \begin{vmatrix} \lambda-1 & 0 & 0 \\ -\frac{1}{2} & \lambda+\frac{1}{2} & -\frac{3}{8} \\ -\frac{1}{2} & 0 & \lambda+\frac{1}{2} \end{vmatrix}$$
$$= (\lambda-1)\begin{vmatrix} \lambda+\frac{1}{2} & -\frac{3}{8} \\ 0 & \lambda+\frac{1}{2} \end{vmatrix} = (\lambda-1)(\lambda+\tfrac{1}{2})^2,$$

which has roots 1 and $-\frac{1}{2}$, so the eigenvalues are 1 and $-\frac{1}{2}$. In the second equality, the second and the third row are added to the first (both operations leave the determinant unchanged), and in the third equality the first column is subtracted from the second and third. Then the determinant is expanded across the first row. Of course, the determinant could have been expanded right away without doing any row or column operations first, but now we got the advantage of avoiding any final factorisation.

The eigenspace of $-\frac{1}{2}$ is the solution set of $(-\frac{1}{2}I - A)\mathbf{x} = \mathbf{0}$. If $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$, we get

$$\begin{aligned} -\tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{8}x_3 &= 0 \\ -\tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 - \tfrac{7}{8}x_3 &= 0 \\ -\tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{2}x_3 &= 0, \end{aligned} \qquad \begin{aligned} x_1 + x_2 + \tfrac{1}{4}x_3 &= 0 \\ x_1 + x_2 + \tfrac{7}{4}x_3 &= 0 \\ x_1 + x_2 \quad + x_3 &= 0, \end{aligned} \qquad \begin{aligned} x_1 + x_2 + \tfrac{1}{4}x_3 &= 0 \\ \tfrac{3}{2}x_3 &= 0 \\ \tfrac{3}{4}x_3 &= 0, \end{aligned}$$

$$\begin{aligned} x_1 + x_2 \quad &= 0 \\ x_3 &= 0, \end{aligned} \qquad \begin{aligned} x_1 &= -x_2 \\ x_3 &= 0, \end{aligned} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -t \\ t \\ 0 \end{bmatrix} = t\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

We have applied elementary row operations to solve the linear system. We could of course have worked with the coefficient matrix instead of working directly with the system. The conclusion is that $\left\{\begin{bmatrix} -1 & 1 & 0 \end{bmatrix}^{\mathrm{T}}\right\}$ is a basis of the eigenspace of $-\frac{1}{2}$.

The eigenspace of 1 is the solution set of $(I - A)\mathbf{x} = \mathbf{0}$. If $\boldsymbol{x} = \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix}^{\mathrm{T}}$, we get

$$\begin{aligned} x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{8}x_3 &= 0 \\ -\tfrac{1}{2}x_1 \quad + x_2 - \tfrac{7}{8}x_3 &= 0 \\ -\tfrac{1}{2}x_1 - \tfrac{1}{2}x_2 \quad + x_3 &= 0, \end{aligned} \qquad \begin{aligned} x_1 - \tfrac{1}{2}x_2 \quad - \tfrac{1}{8}x_3 &= 0 \\ \tfrac{3}{4}x_2 - \tfrac{15}{16}x_3 &= 0 \\ -\tfrac{3}{4}x_2 + \tfrac{15}{16}x_3 &= 0, \end{aligned} \qquad \begin{aligned} x_1 - \tfrac{1}{2}x_2 - \tfrac{1}{8}x_3 &= 0 \\ x_2 - \tfrac{5}{4}x_3 &= 0, \end{aligned}$$

$$\begin{aligned} x_1 \quad - \tfrac{3}{4}x_3 &= 0 \\ x_2 - \tfrac{5}{4}x_3 &= 0, \end{aligned} \qquad \begin{aligned} x_1 &= \tfrac{3}{4}x_3 \\ x_2 &= \tfrac{5}{4}x_3, \end{aligned} \qquad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} \frac{3}{4}s \\ \frac{5}{4}s \\ s \end{bmatrix} = \tfrac{1}{4}s\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix} = t\begin{bmatrix} 3 \\ 5 \\ 4 \end{bmatrix}, \quad s, t \in \mathbb{R}.$$

So $\left\{\begin{bmatrix} 3 & 5 & 4 \end{bmatrix}^{\mathrm{T}}\right\}$ is a basis of the eigenspace of 1.

Since A is a 3×3 matrix and does not have three linearly independent eigenvectors, A is not diagonalisable. (We have found two eigenvectors, which are linearly independent since they correspond to different eigenvalues.)

b) A is a stochastic matrix, since it is square with probablity vectors as columns, that is, vectors with nonnegative entries that add up to 1. A steady-state vector is a probability vector $\mathbf{q}$ such that $A\mathbf{q} = \mathbf{q}$. This implies that $\mathbf{q}$ is an eigenvector corresponding to the eigenvalue 1. From (a), we know that $\mathbf{q} = t\begin{bmatrix}3 & 5 & 4\end{bmatrix}^{\mathrm{T}}$ for a t. To make $\mathbf{q}$ a probability vector, we need $t = 1/(3+5+4) = 1/12$. The steady-state vector for A is thus $\frac{1}{12}\begin{bmatrix}3 & 5 & 4\end{bmatrix}^{\mathrm{T}} = \begin{bmatrix}\frac{1}{4} & \frac{5}{12} & \frac{1}{3}\end{bmatrix}^{\mathrm{T}}$. (The steady-state vector was unique, which also follows from the fact that A is regular – all entries of A^2 are positive.)

Problem 4

First, eigenvectors corresponding to distinct eigenspaces are linearly independent. It is easy to see that the two eigenvectors corresponding to -2 are linearly independent. So the three provided eigenvectors of the 3×3 matrix A are linearly dependent. Then we know from theory that

$$\mathbf{x}(t) = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} e^{-t} + c_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} e^{-2t} + c_3 \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix} e^{-2t},$$

for c_1, c_2 and c_3 in $\mathbb{R}$, is the general solution of the linear system of differential equations.

We want

$$\begin{bmatrix} 3 \\ 3 \\ 2 \end{bmatrix} = \mathbf{x}(0) = c_1 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix}.$$

We solve the linear system by applying elementary row operations to the augmented matrix of the system:

$$\begin{bmatrix} 1 & 3 & 0 & 3 \\ 2 & 0 & 3 & 3 \\ -1 & 1 & -5 & 2 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & -6 & 3 & -3 \\ 0 & 4 & -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 1 \\ 0 & 4 & -5 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & -3 & 3 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & -1 & 1 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 3 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$

The solution of the system is $(c_1, c_2, c_3) = (3, 0, -1)$. So the solution

$$\mathbf{x}(t) = 3 \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} e^{-t} - \begin{bmatrix} 0 \\ 3 \\ -5 \end{bmatrix} e^{-2t}$$

of the system of differential equations has the property $\mathbf{x}(0) = \begin{bmatrix}3 & 3 & 2\end{bmatrix}^{\mathrm{T}}$. Since $\lim_{t\to\infty} e^{-t} = 0$ and $\lim_{t\to\infty} e^{-2t} = 0$, $\lim_{t\to\infty} \mathbf{x}(t) = \begin{bmatrix}0 & 0 & 0\end{bmatrix}^{\mathrm{T}}$.

Problem 5

a)
$$\begin{aligned}\frac{1-e^{i(n+1)\theta}}{1-e^{i\theta}} &= \frac{(1-e^{i(n+1)\theta})e^{-i\theta/2}}{(1-e^{i\theta})e^{-i\theta/2}} = \frac{e^{-i\theta/2}-e^{i(n+1/2)\theta}}{e^{-i\theta/2}-e^{i\theta/2}} \\ &= \frac{\cos\frac{\theta}{2} - i\sin\frac{\theta}{2} - \cos((n+\frac{1}{2})\theta) - i\sin((n+\frac{1}{2})\theta)}{-2i\sin\frac{\theta}{2}} \\ &= \frac{i\cos\frac{\theta}{2} + \sin\frac{\theta}{2} - i\cos((n+\frac{1}{2})\theta) + \sin((n+\frac{1}{2})\theta)}{2\sin\frac{\theta}{2}} \\ &= \frac{1}{2}\left(1+\frac{\sin((n+\frac{1}{2})\theta)}{\sin\frac{\theta}{2}}\right) + i\frac{\cos\frac{\theta}{2} - \cos((n+\frac{1}{2})\theta)}{2\sin\frac{\theta}{2}}\end{aligned}$$

The real part is as stated in the problem.

b) First, $(e^{i\theta})^k = e^{ik\theta} = \cos k\theta + i\sin k\theta$. Inserting $z = e^{i\theta}$ in the provided formula for a finite geometric series, we get $1+\cos\theta+i\sin\theta+\cos 2\theta+i\sin 2\theta+\cdots+\cos n\theta+i\sin n\theta = (1-e^{i(n+1)\theta})/(1-e^{i\theta})$. Taking real parts of both sides of the equality (use (a) for the right hand side), the result follows.

Problem 6

If A is symmetric, there is an orthogonal matrix P and a diagonal matrix D such that $A = PDP^{\mathrm{T}}$. If A is in addition positive definite, all eigenvalues – the entries of the diagonal of D – are positive. Let $D^{1/2}$ denote a diagonal matrix that as entry ii has the square root of the ii entry of D. Then $D^{1/2}D^{1/2} = D$, and $A = PDP^{\mathrm{T}} = PD^{1/2}D^{1/2}P^{\mathrm{T}} = P(D^{1/2})^{\mathrm{T}}D^{1/2}P^{\mathrm{T}} = (D^{1/2}P^{\mathrm{T}})^{\mathrm{T}}(D^{1/2}P^{\mathrm{T}}) = B^{\mathrm{T}}B$, where $B = D^{1/2}P^{\mathrm{T}}$. B is invertible, since P and $D^{1/2}$ are (the latter has n pivot positions). (There even exists a positive definite symmetric matrix B such that $A = B^{\mathrm{T}}B$: We could let $B = PD^{1/2}P^{\mathrm{T}}$.)

Conversely, assume that $A = B^{\mathrm{T}}B$. Then $A^{\mathrm{T}} = (B^{\mathrm{T}}B)^{\mathrm{T}} = B^{\mathrm{T}}(B^{\mathrm{T}})^{\mathrm{T}} = B^{\mathrm{T}}B = A$, so A is symmetric. The quadratic form $\mathbf{x}^{\mathrm{T}}A\mathbf{x} = \mathbf{x}^{\mathrm{T}}B^{\mathrm{T}}B\mathbf{x} = (B\mathbf{x})^{\mathrm{T}}(B\mathbf{x}) = (B\mathbf{x})\cdot(B\mathbf{x})$ is nonnegative, and zero only if $B\mathbf{x} = \mathbf{0}$, by a property of the inner product (dot product). If B is invertible, $B\mathbf{x} = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$, so A is indeed positive definite.