Problem 1 Find all solutions of $-(z+i)^3 = 2+2i$ giving your answers in standard form, and draw them on the complex plane.

Problem 2 Solve the initial value problem:

$$y'' - y' - 6y = te^{3t}, \quad y(0) = 0, y'(0) = 0.$$

Problem 3 Find a particular solution of the ODE:

$$y'' + 2y' + y = t^{-2}e^{-t}.$$

Problem 4 Find all solutions of the following linear system.

$$\begin{aligned} 5x_1 + 10x_2 - 3x_3 + 12x_4 + 8x_5 &= -9 \\ 2x_1 + 4x_2 - x_3 + 5x_4 + x_5 &= 1 \\ x_1 + 2x_2 - x_3 + 2x_4 + x_5 &= -1 \\ -x_1 - 2x_2 \qquad - 3x_4 + 2x_5 &= -6 \end{aligned}$$

Problem 5 Let $V \subseteq \mathbb{R}^4$ be the subspace spanned by

$$\left\{ \begin{bmatrix} 1 \\ 2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

Find the orthogonal projection of $\begin{bmatrix} 4 \\ -2 \\ 0 \\ 2 \end{bmatrix}$ onto V.

Problem 6

a. The matrix $A = \begin{bmatrix} .3 & .6 \\ .7 & .4 \end{bmatrix}$ is a stochastic matrix and so has a steady-state vector which is an eigenvector of A with eigenvalue 1.

Find another eigenvalue of A and its corresponding eigenvector.

b. Let $\vec{q}$ be the steady-state vector for A. Starting with $\vec{v}_0 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and defining $\vec{v}_{k+1} = A\vec{v}_k$, how many iterations are needed to estimate $\vec{q}$ accurate to 2-decimal places?

Your answer does not need to be the minimum number of iterations needed, but must contain an explanation of why the number you give is sufficient.

(Computing some iterations and seeing where it appears to stabilise is not sufficient explanation.)

Problem 7 Find the eigenvalues and eigenvectors of $\begin{bmatrix} -5 & 4 \\ -4 & 5 \end{bmatrix}$ and solve

$$\begin{aligned} x_1' &= -5x_1 + 4x_2 \\ x_2' &= -4x_1 + 5x_2 \end{aligned}$$

with initial conditions $x_1(0) = x_2(0) = 3$.

Problem 8 Using a substitution of the form $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = P \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}$, write the quadratic form $4x_1^2 + 24x_1x_2 + 11x_2^2$ in the form $ay_1^2 + by_2^2$ and sketch the set $\{(x_1, x_2) : 4x_1^2 + 24x_1x_2 + 11x_2^2 = 20\}$.

Problem 9 Let A be an $m \times m$ square matrix. Let λ be an eigenvalue of A. Show that the set $\{\vec{x} \in \mathbb{R}^m : A\vec{x} = \lambda\vec{x}\}$ is a subspace of $\mathbb{R}^m$.