Norwegian University of Science and Technology
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Exam in TMA4110 Calculus 3, June 2013
Solution

Problem 1 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\4\\2\end{bmatrix}, \; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\\1\end{bmatrix}, \; T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\6\\3\end{bmatrix}.$$

a) Find the standard matrix A for the linear transformation T.

Solution. The standard matrix is $A = \left[T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \; T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)\right]$. The first two columns of A are given. To find $T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right)$, we use that T is linear,

$$\begin{aligned} T\left(\begin{bmatrix}0\\0\\1\end{bmatrix}\right) &= T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) - T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) - T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) \\ &= \begin{bmatrix}2\\6\\3\end{bmatrix} - \begin{bmatrix}0\\4\\2\end{bmatrix} - \begin{bmatrix}0\\2\\1\end{bmatrix} = \begin{bmatrix}2\\0\\0\end{bmatrix}. \end{aligned}$$

So $A = \begin{bmatrix}0 & 0 & 2\\4 & 2 & 0\\2 & 1 & 0\end{bmatrix}$.

b) Find a basis for the null space, $\mathrm{Nul}(A)$, of A, and a basis for the column space, $\mathrm{Col}(A)$, of A.

Solution. By Gauss elimination, we see that A is row-equivalent to $B = \begin{bmatrix} 1 & 1/2 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$.

Therefore, the equation $A\mathbf{x} = \mathbf{0}$ has solution $x = t \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, and $\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} \right\}$ is a basis for Nul(A). The pivots of B are in the first and third column, so the first and third column of A, that is $\left\{ \begin{bmatrix} 0 \\ 4 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} \right\}$, is a basis for Col$(A)$.

Problem 2

a) Find the solution of the differential equation $y'' - y' = 0$ which satisfies $y(0) = 1$ and $y'(0) = -1$.

Solution. The characteristic polynomial of the differential equation is $\lambda^2 - \lambda$, with roots $\lambda_1 = 0, \lambda_2 = 1$, so the general solution of the differential equation is

$$y(t) = c_1 e^{0 \cdot t} + c_2 e^{1 \cdot t} = c_1 + c_2 e^t.$$

Enforcing the conditions $y(0) = 1$ and $y'(0) = -1$ gives the linear equations

$$\begin{aligned} c_1 + c_2 &= 1 \\ c_2 &= -1, \end{aligned}$$

Which are solved by $c_1 = 2, c_2 = -1$. The solution is therefore $y(t) = 2 - e^t$.

b) Find the general solution to the differential equation $y'' - y' = e^t \sin t$.

Solution. We must find find a particular solution. There are several ways to proceed here. Undetermined coefficients, variation of parameters, or even setting $x = y'$ and solving the first order equation $x' - x = e^t \sin t$ by using an integrating factor, all lead to a solution. Here we consider the complex differential equation

$$z'' - z' = e^{(1+i)t}. \tag{1}$$

(Notice that $\operatorname{Im}\left(e^{(1+i)t}\right) = e^t \sin t$.) If z_p is a solution of (1), then $y_p = \operatorname{Im}(z_p)$ is a solution of the original equation $y'' - y' = e^t \sin t$. We use the method of undetermined coefficients and look for a solution $z_p = ce^{(1+i)t}$ of (1). Now, $z_p' = (1+i)ce^{(1+i)t}, z_p'' = (1+i)^2 ce^{(1+i)t} = 2ice^{(1+i)t}$, and inserting into (1) gives

$$\begin{aligned} 2ice^{(1+i)t} - (1+i)ce^{(1+i)t} &= e^{(1+i)t}, \\ (-1+i)ce^{(1+i)t} &= e^{(1+i)t}. \end{aligned}$$

For this equality to hold for all t, we need to have $c = \frac{1}{-1+i} = -\frac{1}{2}(1+i)$. Therefore $z_p = -\frac{1}{2}(1+i)e^{(1+i)t}$ solves (1), and

$$\begin{aligned} y_p = \operatorname{Im}(z_p) = -\frac{1}{2}\operatorname{Im}\left((1+i)e^{(1+i)t}\right) &= -\frac{1}{2}\left(\operatorname{Re}(1+i)\operatorname{Im}(e^{(1+i)t}) + \operatorname{Im}(1+i)\operatorname{Re}(e^{(1+i)t})\right) \\ &= -\frac{1}{2}\left(1\cdot e^t \sin t + 1 \cdot e^t \cos t\right) \\ &= -\frac{1}{2}e^t(\sin t + \cos t), \end{aligned}$$

is a partial solution to $y'' - y' = e^t \sin t$.

Alternatively, we can use variation of parameters. We then look for a solution of the form $y_p(t) = v_1(t) + v_2(t)e^t$ (because $y_h(t) = c_1 + c_2e^t$ is the general solution of the homogenous equation $y'' - y' = 0$). We then have $y_p'(t) = v_1'(t) + v_2'(t)e^t + v_2(t)e^t$. Assume that $v_1'(t) + v_2'(t)e^t = 0$. Then $y_p'(t) = v_2(t)e^t$, $y_p''(t) = v_2(t)e^t + v_2'(t)e^t$ and $y_p''(t) - y_p'(t) = v_2'(t)e^t$. The solution of the system

$$\begin{aligned} v_1'(t) + v_2'(t)e^t &= 0 \\ v_2'(t)e^t &= \sin(t)e^t \end{aligned}$$

is $v_2'(t) = \sin(t)$, $v_1'(t) = -\sin(t)e^t$, so if we let $v_2(t) = \int \sin(t)dt = -\cos(t)$ and $v_1(t) = \int -\sin(t)e^t dt = \frac{1}{2}(\cos(t)e^t - \sin(t)e^t)$ (use partial integration or see Rottmann page 144), then $y_p(t) = v_1(t) + v_2(t)e^t = -\frac{1}{2}(\cos(t)e^t + \sin(t)e^t)$ is a partial solution.

We know from **a)** that $y_h(t) = c_1 + c_2e^t$ is the general solution of the homogenous equation $y'' - y' = 0$, so it follows that $y(t) = y_h(t) + y_p(t) = c_1 + c_2e^t - \frac{1}{2}e^t(\sin t + \cos t)$ is the general solution of $y'' - y' = e^t \sin t$.

Problem 3 Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$.

a) Find an orthogonal basis for the plane in $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

Solution. We use the Gram–Schmidt procedure

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{u}_1, \\ \mathbf{w}_2 &= \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \mathbf{u}_2 - \frac{2}{2}\mathbf{w}_1. \end{aligned}$$

Inserting $\mathbf{u}_1$ and $\mathbf{u}_2$ gives $\mathbf{w}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$. $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal basis for the plane spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

b) Find the distance from $\mathbf{v}$ to the plane in $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

Solution. We use the orthogonal basis from **a)** to calculate the orthogonal projection of $\mathbf{v}$ onto $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\} = \text{Span}\{\mathbf{w}_1, \mathbf{w}_2\}$.

$$\hat{\mathbf{v}} = \frac{\mathbf{v}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{v}\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\mathbf{w}_2 = \frac{2}{2}\mathbf{w}_1 + \frac{6}{3}\mathbf{w}_2 = \mathbf{w}_1 + 2\mathbf{w}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$$

The distance from $\mathbf{v}$ to $\text{Span}\{\mathbf{u}_1, \mathbf{u}_2\}$ is given by

$$\|v - \hat{v}\| = \left\| \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix} \right\| = \sqrt{6}.$$

Problem 4 A particle moving in a plane under the influence of a force has the equation of motion

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}(t)$ denotes the position of the particle at the time t. Find $\mathbf{x}(t)$ assuming that $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The answer should be given in the form $\mathbf{x}(t) = e^{at} \begin{bmatrix} c_1\cos(bt) + c_2\sin(bt) \\ c_3\cos(bt) + c_4\sin(bt) \end{bmatrix}$ where a, b, c_1, c_2, c_3 and c_4 are real numbers.

Solution. Let $A = \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix}$. The characteristic polynomial of A is $\lambda^2 + 2\lambda + 5$, with complex roots $\lambda = -1 + 2i$ and $\bar{\lambda} = -1 - 2i$. From the matrix $A - \lambda I = \begin{bmatrix} 1-2i & -5 \\ 1 & -1-2i \end{bmatrix}$, we can see that the eigenvector corresponding to λ is $\mathbf{v} = \begin{bmatrix} 1+2i \\ 1 \end{bmatrix}$. The eigenvector corresponding to $\bar{\lambda}$ is therefore simply $\bar{\mathbf{v}} = \begin{bmatrix} 1-2i \\ 1 \end{bmatrix}$. The general complex solution of $\mathbf{x}'(t) = A\mathbf{x}(t)$ is given in terms of the eigenvectors as

$$\mathbf{x}(t) = c_1 e^{\lambda t}\mathbf{v} + c_2 e^{\bar{\lambda} t}\bar{\mathbf{v}},$$

where c_1 and c_2 are complex numbers. To obtain the solution satisfying the initial condition $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, we get two equations for c_1, c_2.

$$\mathbf{x}(0) = c_1\mathbf{v} + c_2\bar{\mathbf{v}}$$
$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 + 2(c_1 - c_2)i \\ c_1 + c_2 \end{bmatrix}.$$

Subtracting the second equation from the first, we see that $c_1 = c_2$, and then the second equation gives that $c_1 = c_2 = \frac{1}{2}$. The solution of the initial value problem is therefore

$$\mathbf{x}(t) = \frac{1}{2}e^{\lambda t}\mathbf{v} + \frac{1}{2}e^{\bar{\lambda} t}\bar{\mathbf{v}}.$$

To get this expression on the form required, we could expand this expression using $e^{a+bi} = e^a(\cos b + i\sin b)$. A quicker way is to recognise the expression on the right hand side above as $\frac{1}{2}(\mathbf{w}(t) + \bar{\mathbf{w}}(t)) = \mathrm{Re}(\mathbf{w}(t))$ with $\mathbf{w}(t) = e^{\lambda t}\mathbf{v}$. So

$$\begin{aligned}
\mathbf{x}(t) &= \mathrm{Re}\left(e^{\lambda t}\mathbf{v}\right) \\
&= \mathrm{Re}\left(e^{(-1+2i)t}\begin{bmatrix}1+2i \\ 1\end{bmatrix}\right) \\
&= e^{-t}\,\mathrm{Re}\left(\begin{bmatrix}e^{2it}(1+2i) \\ e^{2it}\end{bmatrix}\right) \\
&= e^{-t}\,\mathrm{Re}\left(\begin{bmatrix}(\cos(2t)+i\sin(2t))(1+2i) \\ \cos(2t)+i\sin(2t)\end{bmatrix}\right) \\
&= e^{-t}\,\mathrm{Re}\left(\begin{bmatrix}\cos(2t)-2\sin(2t)+i(2\cos(2t)+\sin(2t)) \\ \cos(2t)+i\sin(2t)\end{bmatrix}\right) \\
&= e^{-t}\begin{bmatrix}\cos(2t)-2\sin(2t) \\ \cos(2t)\end{bmatrix}
\end{aligned}$$

Problem 5 You are given that

$$\det\left(\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}\right) = 2.$$

Use this information to compute the determinant of the matrix

$$\begin{bmatrix} x & y & z \\ p & q & r \\ 5p-2a & 5q-2b & 5r-2c \end{bmatrix}.$$

Give reasons for your answer.

Solution. By the properties of determinants and elementary row operations

$$\begin{aligned}
\det\left(\begin{bmatrix} x & y & z \\ p & q & r \\ 5p-2a & 5q-2b & 5r-2c \end{bmatrix}\right) &= -\det\left(\begin{bmatrix} 5p-2a & 5q-2b & 5r-2c \\ p & q & r \\ x & y & z \end{bmatrix}\right), \\
&= -\det\left(\begin{bmatrix} -2a & -2b & -2c \\ p & q & r \\ x & y & z \end{bmatrix}\right), \\
&= -(-2)\det\left(\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}\right), \\
&= 2 \cdot 2 = 4.
\end{aligned}$$

Problem 6 *You do not have to give reasons for your answers for this problem.*

a) For each of the following 4 complex numbers, determine whether it lies in the first quadrant of the complex plane (i.e., both its real part and its imaginary part are non-negative) or not.

1. $\sqrt{3} - i$.
   Not in first quadrant.
2. $\frac{-2+i}{2+3i}$.
   Not in first quadrant. The expression is equal to $-\frac{1}{13} + \frac{8}{13}i$.
3. $e^{-2+7\pi i}$.
   Not in first quadrant. $e^{-2+7\pi i} = e^{-2}e^{7\pi i} = e^{-2}e^{\pi i} = -e^2$.
4. z^2, where $|z| = 2$ and $\text{Arg}(z) = \frac{\pi}{3}$.
   Not in first quadrant. $\arg(z^2) = 2 \cdot \text{Arg}(z) = \frac{2\pi}{3} > \frac{\pi}{2}$

b) Let A be an $n \times n$ matrix, B an $m \times n$ matrix, and C an $n \times m$ matrix, where $n \neq m$. For each of the following 4 expressions, determine whether it is well defined or not.

1. AB^T.
   Defined.
2. BB^T.
   Defined.
3. $CB + 2A$.
   Defined.

4. $B^2 - A^2$.
   Not defined. B^2 is not defined.

c) Let A and D be $n \times n$ matrices and let $\mathbf{b}$ be a nonzero vector in $\mathbb{R}^n$. For each of the following 4 statements, determine whether it is true or not.

1. If the system $A\mathbf{x} = \mathbf{b}$ has more than one solution, then the system $A\mathbf{x} = 0$ also has more than one solution.
   True. If $\mathbf{x}_1$ and $\mathbf{x}_2$ both solves $A\mathbf{x} = \mathbf{b}$, then $A(\mathbf{x}_1 - \mathbf{x}_2) = A\mathbf{x}_1 - A\mathbf{x}_2 = \mathbf{b} - \mathbf{b} = \mathbf{0}$.
2. If A^T is non-invertible, then A is non-invertible.
   True. A^T non-invertible $\Leftrightarrow$ $\text{rank}(A^T) < n \Leftrightarrow \text{rank}(A) < n \Leftrightarrow A$ non-invertible.
3. If $AD = I$, then $DA = I$.
   True. A and D are square, so if $AD = I$ then $D = A^{-1}$.
4. If A has orthonormal columns, then A is invertible.
   True. If A has orthonormal columns, then $A^T A = I$, so $A^T = A^{-1}$.

d) For each of the following 4 statements, determine whether it is true or not.

1. The two vectors $\begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ are orthogonal.
   Not true.
2. If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$, then $\mathbf{z}$ belongs to the orthogonal complement of $\text{Span}\{\mathbf{x}, \mathbf{y}\}$.
   True. $\mathbf{z}$ is orthogonal to both $\mathbf{x}$ and $\mathbf{y}$, and thus to all of $\text{Span}\{\mathbf{x}, \mathbf{y}\}$.
3. An $m \times n$ matrix B has orthonormal columns if and only if $BB^T = I$.
   Not true. (B has orthonormal columns if and only if $B^T B = I$.) $BB^T = I$ is equivalent to A having orthonormal *rows*, but nonsquare matrices may have orthonormal rows but not orthonormal columns and vice versa.
4. If $\mathbf{x}$ is orthogonal to $\mathbf{y}$ and $\mathbf{z}$, then $\mathbf{x}$ is orthogonal to $\mathbf{y} - \mathbf{z}$.
   True. $\mathbf{x}^T(\mathbf{y} - \mathbf{z}) = \mathbf{x}^T\mathbf{y} - \mathbf{x}^T\mathbf{z} = 0$

e) Let A be an $n \times n$ matrix. For each of the following 4 statements, determine whether it is true or not.

1. If A is orthogonally diagonalizable, then A is symmetric.
   True. $A = PDP^T \Leftrightarrow A^T = (P^T)^T D^T P^T = PDP^T = A$.

2. If A is an orthogonal matrix, then A is symmetric.
   Not true. Counterexample: $A = \left[\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right]$ is orthogonal, but not symmetric.
3. If $\mathbf{x}^T A\mathbf{x} > 0$ for every $\mathbf{x} \neq \mathbf{0}$, then the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite.
   True. By definition.
4. Every quadratic form can by a change of variable be transformed into a quadratic form with no cross-product term.
   True. Every quadratic form can be written $\mathbf{x}^T A\mathbf{x}$ with A symmetric. When A is symmetric, it is also orthogonally diagonalizable, $A = PDP^T$, so the change of variables defined by $\mathbf{x} = P\mathbf{y}$ gives $\mathbf{x}^T A\mathbf{x} = \mathbf{y}^T D\mathbf{y}$.

f) Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{R}^n$, and let r be a scalar. For each of the following 4 statements, determine whether it is true or not.

1. $\|r\mathbf{v}\| = r\|\mathbf{v}\|$, unless $r = 0$.
   Not true. Does not hold for $r < 0$..
2. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.
   True. If $a\mathbf{u} + b\mathbf{v} = 0$, then $\mathbf{u}^T(a\mathbf{u} + b\mathbf{v}) = a\mathbf{u}^T\mathbf{u} + b\mathbf{u}^T\mathbf{v} = a\|u\|^2 = 0$, so $a = 0$. Similarly, $\mathbf{v}^T(a\mathbf{u} + b\mathbf{v}) = 0$ gives $b = 0$.
3. If $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   True. In general, $\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v})^T(\mathbf{u} + \mathbf{v}) = \|\mathbf{u}\|^2 + 2\mathbf{u}^T\mathbf{v} + \|\mathbf{v}\|^2$ holds.
4. If $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
   True. This is the same as 3 where $-\mathbf{v}$ is substituted for $\mathbf{v}$.

g) For each of the following 4 statements, determine whether it is true or not.

1. If A is a matrix, then $\text{rank}(A) = \dim(\text{Nul}(A))$.
   Not true. By the definition of rank, $\text{rank}(A) = \dim(\text{Row}(A)) = \dim(\text{Col}(A))$, $\dim(\text{Nul}(A))$ is usually different.
2. A 5×10 matrix can have a 2-dimensional null space.
   Not true. A 5×10 matrix has maximal rank 5. The rank theorem tells us that the rank of the matrix plus the dimension of the null space for a 5×10 matrix is equal to 10. Therefore the dimension of the null space is at least 5.
3. Row operations on a matrix can change its null space.
   Not true. This is a basic property of row operations.
4. If the matrices A and B have the same reduced echelon form, then $\text{Row}(A) = \text{Row}(B)$.
   True. The row space of a matrix is spanned by the nonzero rows of its reduced echelon form.

h) For each of the following 4 statements, determine whether it is true or not.

1. The polynomials $p_1(t) = 1 + t^2$ and $p_2(t) = 1 - t^2$ are linearly independent.
   True. $p_2(t)/p_1(t)$ is defined for all t, but not constant.
2. If A is a 3×4 matrix, then the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$.
   Not true. The mapping is a linear transformation from $\mathbb{R}^4$ to $\mathbb{R}^3$.
3. If a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ is onto $\mathbb{R}^4$ (or is surjective), then T cannot be one-to-one (injective).
   Not true. A linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ which is surjective is in fact invertible, and therefore also injective.
4. A linear transformation $S : \mathbb{R}^5 \to \mathbb{R}^4$ cannot be one-to-one (injective).
   True. The null space of the transformation has to be at least one-dimensional, so $S(\mathbf{x}) = \mathbf{0}$ has infinitely many solutions.