Norwegian University of Science and Technology
Department of Mathematical Sciences

Exam in TMA4110 Calculus 3, December 2012

Solutions

Problem 1 Show that $z_1 = 1 + \sqrt{3}i$ is a zero of the polynomial $P(z) = z^5 - 2z^4 + 4z^3 - 8z^2 + 16z - 32$ and find the 4 other zeros of P.

Solution. The best strategy here is to recall that z_0 is a zero of a polynomial $Q(z)$ with real coefficients if and only if $\overline{z_0}$ is a zero of $Q(z)$. Hence, since $P(z)$ has real coefficients, we know that z_1 is a zero if and only if $z_2 = \overline{z_1} = 1 - \sqrt{3}i$ is a zero. Showing that z_1 is a zero is then the same as showing that both z_1 and z_2 are zeros. This happens if and only if

$$(z - z_1)(z - z_2) = z^2 - 2z + 4$$

is a factor of $P(z)$. Performing the division $P(z) : (z^2 - 2z + 4)$, we get that $P(z) = (z^2 - 2z + 4)(z^3 - 8)$, so z_1 and z_2 are zeros.

The remaining zeros are the three complex numbers satisfying $z^3 = 8$. We use polar form for the calculations. We have that $8 = 8(\cos 0 + i \sin 0)$. Let $z = r(\cos\theta + i\sin\theta)$. Then we know that $r = \sqrt[3]{8} = 2$, and that

$$\theta = \frac{0 + 2k\pi}{3} \quad \text{for } k \in \{0, 1, 2\},$$

so the three arguments are $\theta = 0$, $\theta = \frac{2\pi}{3}$ and $\theta = \frac{4\pi}{3}$. This gives the three solutions

$$\begin{aligned}
z_3 &= 2(\cos 0 + i \sin 0) = 2 \\
z_4 &= 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right) = -1 + \sqrt{3}i \\
z_5 &= 2\left(\cos\frac{4\pi}{3} + i\sin\frac{4\pi}{3}\right) = -1 - \sqrt{3}i
\end{aligned}$$

Note 1: Instead of computing z_5, you can use the fact that $z_5 = \overline{z_4}$.

Note 2: If you missed the conjugation trick in the first part, it is also possible to evaluate $P(z_1)$. Then you will have to compute z_1^2, z_1^3 and so on. Fortunately, $z_1^3 = -8$, so the computations won't be too hard. It is also possible (but not preferable) to perform the division $P(z) : (z - z_1)$. In both of these cases, finding the other 4 roots is difficult.

Problem 2 Find the general solution to the differential equation $y'' + 2y' + 5y = 2\cos t + 4\sin t$.

Solution. The general solution is of the form $y = y_h + y_p$, where y_h is the general solution of the homogeneous equation $y'' + 2y' + 5y = 0$, and y_p is some particular solution of the original equation. For y_h, we consider the characteristic polynomial $\lambda^2 + 2\lambda + 5 = 0$. We get two complex roots: $\lambda = -1 \pm 2i$. Thus, $y_h = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t$.

As for the particular solution y_p, we have several methods which can be applied. Here, we guess that the solution is of the form $y(t) = A\cos t + B\sin t$ for some real numbers A and B. Then

$$\begin{aligned} y' &= -A\sin t + B\cos t \\ y'' &= -A\cos t - B\sin t \end{aligned}$$

Inserted into the equation, we get

$$\begin{aligned} (-A\cos t - B\sin t) + 2(-A\sin t + B\cos t) + 5(A\cos t + B\sin t) &= 2\cos t + 4\sin t \\ (4A + 2B)\cos t + (4B - 2A)\sin t &= 2\cos t + 4\sin t \end{aligned}$$

So the equations we need to solve is $4A+2B = 2$ and $4B-2A = 4$. The solution is $A = 0, B = 1$. Hence, the particular solution is $y_p = \sin t$. The general solution is

$$y(t) = c_1 e^{-t}\cos 2t + c_2 e^{-t}\sin 2t + \sin t$$

Problem 3 Find the general solution to the system

$$\begin{aligned} 3x_1 - 6x_2 + 6x_3 &= -15 \\ x_1 + x_2 \ + 4x_3 &= 10. \end{aligned}$$

Solution. We reduce the augmented matrix of the system:

$$\begin{bmatrix} 3 & -6 & 6 & -15 \\ 1 & 1 & 4 & 10 \end{bmatrix} \sim \begin{bmatrix} 0 & -9 & -6 & -45 \\ 1 & 1 & 4 & 10 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 4 & 10 \\ 0 & 1 & 2/3 & 5 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 10/3 & 5 \\ 0 & 1 & 2/3 & 5 \end{bmatrix}$$

x_3 is free, so we put $x_3 = s$. Then $x_1 = 5 - \frac{10}{3}s$ and $x_2 = 5 - \frac{2}{3}s$, and the general solution is

$$x = \begin{bmatrix} 5 - 10/3s \\ 5 - 2/3s \\ s \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix} + s \begin{bmatrix} -10/3 \\ -2/3 \\ 1 \end{bmatrix}$$

Problem 4 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be an invertible linear transformation such that $T(x_1, x_2, x_3) = (x_2 + 2x_3, x_1 + 3x_3, 4x_1 - 3x_2 + 8x_3)$. Find a formula for T^{-1}.

Solution. First, we find the standard matrix A of T. This is $\begin{bmatrix} T(\mathbf{e}_1) & T(\mathbf{e}_2) & T(\mathbf{e}_3) \end{bmatrix}$, where $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is the standard basis of $\mathbb{R}^3$. It is easy to see/compute that

$$A = \begin{bmatrix} 0 & 1 & 2 \\ 1 & 0 & 3 \\ 4 & -3 & 8 \end{bmatrix}.$$

To find a formula for T^{-1}, we find A^{-1}. This is done by reducing

$$\begin{bmatrix} 0 & 1 & 2 & 1 & 0 & 0 \\ 1 & 0 & 3 & 0 & 1 & 0 \\ 4 & -3 & 8 & 0 & 0 & 1 \end{bmatrix}$$

to

$$\begin{bmatrix} 1 & 0 & 0 & -9/2 & 7 & -3/2 \\ 0 & 1 & 0 & -2 & 4 & -1 \\ 0 & 0 & 1 & 3/2 & -2 & 1/2 \end{bmatrix}.$$

So

$$T(x_1, x_2, x_3) = A^{-1} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -9/2 & 7 & -3/2 \\ -2 & 4 & -1 \\ 3/2 & -2 & 1/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -(9/2)x_1 + 7x_2 - (3/2)x_3 \\ -2x_1 + 4x_2 - x_3 \\ (3/2)x_1 - 2x_2 + (1/2)x_3 \end{bmatrix},$$

and the formula for T^{-1} is $T^{-1}(x_1, x_2, x_3) = (-\frac{9}{2}x_1 + 7x_2 - \frac{3}{2}x_3, -2x_1 + 4x_2 - x_3, \frac{3}{2}x_1 - 2x_2 + \frac{1}{2}x_3)$.

Problem 5 Let $A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix}$. Find orthonormal bases for $\mathrm{Col}(A)$, $\mathrm{Row}(A)$, and $\mathrm{Nul}(A)$.

Solution. First, we reduce A to reduced echelon form:

$$A = \begin{bmatrix} 1 & 1 & 2 \\ 2 & 2 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = A'$$

Now, A' give us all the information we need about **bases** (not necessarily orthonormal) for Col(A), Row(A) and Nul(A):

A basis for Nul(A) is found by solving $A'\mathbf{x} = \mathbf{0}$. We get that x_2 and x_3 are free, so we put $x_2 = s$ and $x_3 = t$. Then $x_1 = -2t - s$, and the general solution is $\mathbf{x} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} s + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} t$. Hence, a basis for the null space is $\{\mathbf{v}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}\}$. It is neither orthogonal nor orthonormal. We orthogonalize it by using the Gram-Schmidt algorithm:

First, we put $\mathbf{u}_1 = \mathbf{v}_1$. Then, we compute

$$\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2\mathbf{u}_1}{\mathbf{u}_1\mathbf{u}_1}\mathbf{u}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix}$$

Finally, to get the basis orthonormal, we normalize each vector – that is, we divide each entry by the norm of the vector.

$$\mathbf{u}_1' = \frac{1}{\|\mathbf{u}_1\|}\mathbf{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

$$\mathbf{u}_2' = \frac{1}{\|\mathbf{u}_2\|}\mathbf{u}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{3} \\ -1/\sqrt{3} \\ 1/\sqrt{3} \end{bmatrix}$$

So $\{\mathbf{u}_1', \mathbf{u}_2'\}$ is an orthonormal basis for Nul(A).

A basis for Row(A) can be read directly from A', picking the rows with a pivot. In this case, there is only one such row, so the basis is $\{\mathbf{u}_3 = \begin{bmatrix} 1 & 1 & 2 \end{bmatrix}\}$. Since the basis consists of only one vector, it is orthogonal, but we need to normalize it:

$$\mathbf{u}_3' = \frac{1}{\|\mathbf{u}_3\|}\mathbf{u}_1 = \begin{bmatrix} 1/\sqrt{6} & 1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}$$

So $\{\mathbf{u}_3'\}$ is a basis for Row(A).

For the column space, we pick the columns of A with a corresponding pivot in A'. Hence a basis for Col(A) is $\{\mathbf{u}_4 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$. Again, this is an orthogonal set, but we need to normalize. The norm of $\mathbf{u}_4$ is $\sqrt{5}$, so an orthonormal basis for Col(A) is $\{\mathbf{u}_4' = \begin{bmatrix} 1/\sqrt{5} \\ 2/\sqrt{5} \end{bmatrix}\}$.

Problem 6 Let $P = \begin{bmatrix} 0.8 & 0.3 \\ 0.2 & 0.7 \end{bmatrix}$. Let $\mathbf{x}_0, \mathbf{x}_1, \mathbf{x}_2, \ldots$ be the Markov chain defined by $\mathbf{x}_0 = \begin{bmatrix} 0.4 \\ 0.6 \end{bmatrix}$ and $\mathbf{x}_{i+1} = P\mathbf{x}_i$ for $i = 0, 1, 2, \ldots$.

Find the steady-state vector for P and an explicit formula for $\mathbf{x}_i$.

Solution. The steady-state vector $\mathbf{q}$ is a vector satisfying $P\mathbf{q} = \mathbf{q}$, and with entries adding up to 1 (note that this makes sense only if 1 is an eigenvalue of P, and then $\mathbf{q}$ is a corresponding eigenvector). Since we are going to need all eigenvalues later (to find the explicit formula for $\mathbf{x}_i$), we consider the characterisic equation of P.

$$\det(P - I\lambda) = \det \begin{bmatrix} 0.8-\lambda & 0.3 \\ 0.2 & 0.7-\lambda \end{bmatrix} = \lambda^2 - 1.5\lambda + 0.5 = (\lambda - 1)(\lambda - 0.5)$$

Let $\lambda_1 = 1$ and $\lambda_2 = 0.5$. We solve $(P - I\lambda_1)\mathbf{x} = \mathbf{0}$ to find a basis for the eigenspace corresponding to λ_1. We get

$$\begin{bmatrix} -0.2 & 0.3 \\ 0.2 & -0.3 \end{bmatrix} \sim \begin{bmatrix} 1 & -3/2 \\ 0 & 0 \end{bmatrix}$$

So $\{\mathbf{x} = \left[\begin{smallmatrix} 3/2 \\ 1 \end{smallmatrix}\right]\}$ is a basis for the eigenspace. We let $\mathbf{q} = \frac{1}{5/2}\mathbf{x} = \begin{bmatrix} 3/5 \\ 2/5 \end{bmatrix}$. This is the steady-state vector.

To find the explicit formula for $\mathbf{x}_i$, we need one eigenvector corresponding to each eigenvalue. For λ_1, we pick $\mathbf{u}_1 = \left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix}\right]$ (multiply $\mathbf{q}$ by 5 to get rid of the fractions). For λ_2, we find that $\mathbf{u}_2 = \left[\begin{smallmatrix} -1 \\ 1 \end{smallmatrix}\right]$ is an eigenvector (by solving $(P - I\lambda_2)\mathbf{x} = \mathbf{0}$).

The next ting we do, is to find real numbers A and B such that $\mathbf{x}_0 = A\mathbf{u}_1 + B\mathbf{u}_2$ (note that such numbers must exist; since $\mathbf{u}_1$ and $\mathbf{u}_2$ are linearly independent, they form a basis for $\mathbb{R}^2$). The augmented matrix of this equation is

$$\begin{bmatrix} 3 & -1 & 0.4 \\ 2 & 1 & 0.6 \end{bmatrix}.$$

When solving this system, we get $A = B = \frac{1}{5}$.

Now, $\mathbf{x}_1 = P\mathbf{x}_0 = P(\frac{1}{5}\mathbf{u}_1 + \frac{1}{5}\mathbf{u}_2) = \frac{1}{5}P\mathbf{u}_1 + \frac{1}{5}P\mathbf{u}_2 = \frac{1}{5}\lambda_1\mathbf{u}_1 + \frac{1}{5}\lambda_2\mathbf{u}_2$ (recall that $P\mathbf{u}_i = \lambda_i\mathbf{u}_i$). Similarly, $\mathbf{x}_2 = P\mathbf{x}_1 = P(P\mathbf{x}_0) = P(\frac{1}{5}\lambda_1\mathbf{u}_1 + \frac{1}{5}\lambda_2\mathbf{u}_2) = \frac{1}{5}\lambda_1 P\mathbf{u}_1 + \frac{1}{5}\lambda_2 P\mathbf{u}_2 = \frac{1}{5}\lambda_1^2\mathbf{u}_1 + \frac{1}{5}\lambda_2^2\mathbf{u}_2$. Continuing this way, we get the formula

$$\mathbf{x}_i = \frac{1}{5}\lambda_1^i\mathbf{u}_1 + \frac{1}{5}\lambda_2^i\mathbf{u}_2 = \frac{1}{5}1^i \begin{bmatrix} 3 \\ 2 \end{bmatrix} + \frac{1}{5}0.5^i \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

for $\mathbf{x}_i$.

Problem 7 Find the solution of the system

$$\begin{aligned} x_1' &= x_1 + 3x_2 + 3x_3 \\ x_2' &= -3x_1 - 5x_2 - 3x_3 \\ x_3' &= 3x_1 + 3x_2 + x_3 \end{aligned}$$

that satisfies $x_1(0) = 1$, $x_2(0) = -1$ and $x_3(0) = 2$.

Solution. The system can be written in matrix form like $\mathbf{x}' = A\mathbf{x}$, where

$$A = \begin{bmatrix} 1 & 3 & 3 \\ -3 & -5 & -3 \\ 3 & 3 & 1 \end{bmatrix}.$$

We want to find a matrix P which diagonalizes A. We study the characteristic equation:

$$\det(A - I\lambda) = \det \begin{bmatrix} 1-\lambda & 3 & 3 \\ -3 & -5-\lambda & -3 \\ 3 & 3 & 1-\lambda \end{bmatrix} = -(\lambda - 1)(\lambda + 2)^2$$

We find bases for the eigenspaces corresponding to $\lambda_1 = 1$ and $\lambda_2 = -2$: For λ_1, we get that the eigenspace is spanned by $\left[\begin{smallmatrix} 1 \\ -1 \\ 1 \end{smallmatrix}\right]$; and for λ_2, we get that the eigenspace is spanned by $\left[\begin{smallmatrix} -1 \\ 0 \\ 1 \end{smallmatrix}\right]$ and $\left[\begin{smallmatrix} -1 \\ 1 \\ 0 \end{smallmatrix}\right]$. Hence, P is

$$P = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

We know that $P^{-1}AP = D$, where D is a diagonal matrix with the eigenvalues on the diagonal. Now we can return to our system of differential equations. We will use the substitution $\mathbf{x} = P\mathbf{y}$. Then we get

$$\begin{aligned} P\mathbf{y}' &= AP\mathbf{y} \\ \mathbf{y}' &= P^{-1}AP\mathbf{y} = D\mathbf{y} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -2 \end{bmatrix} \mathbf{y} \\ \mathbf{y} &= \begin{bmatrix} c_1 e^t \\ c_2 e^{-2t} \\ c_3 e^{-2t} \end{bmatrix} \end{aligned}$$

This gives that

$$\mathbf{x} = P\mathbf{y} = \begin{bmatrix} 1 & -1 & -1 \\ -1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} c_1 e^t \\ c_2 e^{-2t} \\ c_3 e^{-2t} \end{bmatrix} = \begin{bmatrix} c_1 e^t - c_2 e^{-2t} - c_3 e^{-2t} \\ -c_1 e^t + c_3 e^{-2t} \\ c_1 e^t + c_2 e^{-2t} \end{bmatrix}$$

We have that $x_1(0) = 1$, $x_2(0) = -1$ and $x_3(0) = 2$. This gives the equations

$$c_1 - c_2 - c_3 = 1$$
$$-c_1 + c_3 = -1$$
$$c_1 + c_2 = 2$$

Using Gauss-Jordan elimination, we get that $c_1 = 2$, $c_2 = 0$ and $c_3 = 1$. Thus,

$$\begin{aligned} x_1(t) &= 2e^t - e^{-2t} \\ x_2(t) &= -2e^t + e^{-2t} \\ x_3(t) &= 2e^t \end{aligned}$$

is the solution satisfying the given initial conditions.

Problem 8 Find the equation $y = \beta_0 + \beta_1 x$ of the least-squares line that best fits the data points (1,3), (2,5), (4,7) and (5,9).

Solution. We will solve the problem by solving the normal equations, but to find the normal equations, we need to express the problem in terms of a matrix equation. First, we form the design matrix

$$X = \begin{bmatrix} 1 & 1 \\ 1 & 2 \\ 1 & 4 \\ 1 & 5 \end{bmatrix}$$

and the observation vector

$$\mathbf{y} = \begin{bmatrix} 3 \\ 5 \\ 7 \\ 9 \end{bmatrix}$$

Now, we can express the problem as: Find the least-squares solution of $X\boldsymbol{\beta} = \mathbf{y}$. The normal equations are $X^T X\boldsymbol{\beta} = X^T\mathbf{y}$. We compute

$$X^T X = \begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix}$$

and

$$X^T\mathbf{y} = \begin{bmatrix} 24 \\ 86 \end{bmatrix}$$

so the system we need to solve is

$$\begin{bmatrix} 4 & 12 \\ 12 & 46 \end{bmatrix} \begin{bmatrix} \beta_0 \\ \beta_1 \end{bmatrix} = \begin{bmatrix} 24 \\ 86 \end{bmatrix}$$

The solution is

$$\beta_0 = \frac{18}{10} \quad \text{and} \quad \beta_1 = \frac{14}{10}$$

so the best-fitting line is $y = \frac{18}{10} + \frac{14}{10}x$.