Norges teknisk–
naturvitenskapelige universitet
Institutt for matematiske fag

LØSNINGSFORSLAG TIL EKSAMEN I FAG SIF5009 MATEMATIKK 3

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Oppgave 1

a) $r^2 + r - 2 = (r-1)(r-2) = 0 \Rightarrow r = 1, -2$

$$\begin{array}{r|l}
-2 & y = Axe^x + Bx + C \quad \text{(partikulær løsning)} \\
1 & y' = Axe^x + Ae^x + B \\
1 & y'' = Axe^x + 2Ae^x \\
\hline
\end{array}$$

$3e^x - 4x = 3Ae^x - 2Bx + B - 2C \Rightarrow A = 1, B = 2, C = 1$

$\underline{y = c_1 e^x + c_2 e^{-2x} + xe^x + 2x + 1}$

b) $r^3 - r^2 + 4r - 4 = (r-1)(r^2+4) = 0 \Rightarrow r = 1, \pm 2i$

$$\begin{array}{r|rll}
-4 & y = & A\cos x & +B\sin x \qquad \text{(partikuær løsning)} \\
4 & y' = & B\cos x & -A\sin x \\
-1 & y'' = & -A\cos x & -B\sin x \\
1 & y''' = & -B\cos x & +A\sin x \\
\hline
\end{array}$$

$\cos x = (-3A + 3B)\cos x + (-3A - 3B)\sin x \Rightarrow$

$$\left.\begin{array}{r} -3A + 3B = 1 \\ -3A - 3B = 0 \end{array}\right\} \Rightarrow A = -\tfrac{1}{6},\ B = \tfrac{1}{6}$$

$\underline{y = c_1 e^x + c_2 \cos 2x + c_3 \sin 2x + \frac{1}{6}(\sin x - \cos x)}$

c) Vi må ha $r^3 + ar^2 + br - 2 = (r-1)(r+2)(r+\alpha)$ der α må være 1 pga konstantleddet, dvs.

$(r-1)(r+2)(r+1) = r^3 + 2r^2 - r - 2$. Så $a = 2$, $b = -1$

Oppgave 2

a) $$\left[\begin{array}{rrrrr|l} -3 & 6 & -8 & -13 & 7 & a \\ 1 & -2 & 2 & 3 & -1 & b \\ 2 & -4 & 5 & 8 & -4 & c \end{array}\right] \sim \left[\begin{array}{rrrrr|l} 1 & -2 & 2 & 3 & -1 & b \\ -3 & 6 & -8 & -13 & 7 & a \\ 2 & -4 & 5 & 8 & -4 & c \end{array}\right]$$

$$\left[\begin{array}{rrrrr|l} 1 & -2 & 2 & 3 & -1 & b \\ 0 & 0 & -2 & -4 & 4 & a+3b \\ 0 & 0 & 1 & 2 & -2 & c-2b \end{array}\right] \sim \left[\begin{array}{rrrrr|l} 1 & -2 & 2 & 3 & -1 & b \\ 0 & 0 & 1 & 2 & -2 & c-2b \\ 0 & 0 & -2 & -4 & 4 & a+3b \end{array}\right] \sim$$

$$\left[\begin{array}{rrrrr|l} 1 & -2 & 2 & 3 & -1 & b \\ 0 & 0 & 1 & 2 & -2 & c-2b \\ 0 & 0 & 0 & 0 & 0 & a-b+2c \end{array}\right]. \ (*) \text{ har løsning} \Leftrightarrow \underline{a-b+2c=0}$$

b) $$\left[\begin{array}{rrrrr|r} 1 & -1 & 2 & 3 & -1 & 2 \\ 0 & 0 & 1 & 2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right] \sim \left[\begin{array}{rrrrr|r} 1 & -2 & 0 & -1 & 3 & 4 \\ 0 & 0 & 1 & 2 & -2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right], \quad \begin{array}{l} x_2 = r \\ x_4 = s \\ x_5 = t \end{array}$$

$$x = \begin{bmatrix} 2r + s - 3t + 4 \\ r \\ -2s + 2t - 1 \\ s \\ t \end{bmatrix} = \underline{r\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 4 \\ 0 \\ -1 \\ 0 \\ 0 \end{bmatrix}; \quad r,s,t \in \mathbb{R}}$$

Basis Null (A): $\underline{\begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}}$ (f.eks.)

c) Basis Row (A): $\underline{\begin{bmatrix} 1 \\ -2 \\ 0 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 2 \\ -2 \end{bmatrix}}$ (f.eks.)

Basis Col (A): $\underline{\begin{bmatrix} -3 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -8 \\ 2 \\ 5 \end{bmatrix}}$ (f.eks.)

d) a) gir at $\underline{\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}}$ er en basis for Col $(A)^{\perp}$.

Oppgave 3

a) $$\det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 & -1 \\ 1 & 1-\lambda & 1 \\ 1 & 1 & 1-\lambda \end{vmatrix} = \cdots = -\lambda(\lambda-1)(\lambda-2) = 0 \Rightarrow \underline{\lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2}$$

$$\lambda_1 = 0 : A - \lambda I = \begin{bmatrix} 1 & 1 & -1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{x_1 = t \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \ t \neq 0}$$

$$\lambda_2 = 1 : A - \lambda I = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{x_2 = t \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}, \ t \neq 0}$$

$$\lambda_3 = 2 : A - \lambda I = \begin{bmatrix} -1 & 1 & -1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{bmatrix} \sim \cdots \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \Rightarrow \underline{x_3 = t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \ t \neq 0}$$

b) $$\underline{P = \begin{bmatrix} 1 & 1 & 0 \\ -1 & -1 & 1 \\ 0 & -1 & 1 \end{bmatrix}, \quad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}}$$

c) $$\underline{x = c_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^t \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + c_3 e^{2t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}; \quad c_1, c_2, c_3 \in \mathbb{R}}$$

Oppgave 4

a) $$u_1 = v_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$

$$u_2 = v_2 - \frac{v_2 \cdot u_1}{u_1 \cdot u_1} u_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \\ 1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}$$

$$u_3 = v_3 - \frac{v_3 \cdot u_1}{u_1 \cdot u_1} u_1 - \frac{v_3 \cdot u_2}{u_2 \cdot u_2} u_2 = \begin{bmatrix} 1 \\ 3 \\ 5 \\ -1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix} - \frac{3}{3} \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 4 \\ -2 \end{bmatrix}$$

$$\text{Ortogonal basis for } V: \underline{\begin{bmatrix}1\\1\\0\\1\end{bmatrix}, \begin{bmatrix}1\\-1\\1\\0\end{bmatrix}, \begin{bmatrix}-1\\3\\4\\-2\end{bmatrix}}, \quad \text{(f.eks.)}$$

b) $$\text{proj}_V\, b = \frac{b\cdot u_1}{u_1\cdot u_1}u_1 + \frac{b\cdot u_2}{u_2\cdot u_2}u_2 + \frac{b\cdot u_3}{u_3\cdot u_3}u_3 = \frac{9}{3}\begin{bmatrix}1\\1\\0\\1\end{bmatrix} + \frac{3}{3}\begin{bmatrix}1\\-1\\1\\0\end{bmatrix} + \frac{30}{30}\begin{bmatrix}-1\\3\\4\\-2\end{bmatrix} = \underline{\begin{bmatrix}3\\5\\5\\1\end{bmatrix}}$$

Oppgave 5

$$P = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix},\ P^{-1} = \begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix},\ D = \begin{bmatrix}3 & 0\\0 & 1\end{bmatrix} \quad \text{gir}$$

$$A = PDP^{-1} = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}\begin{bmatrix}3 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}1 & -1\\-1 & 2\end{bmatrix} = \underline{\begin{bmatrix}5 & -4\\2 & -1\end{bmatrix}}.$$

Oppgave 6

$$\left.\begin{aligned} x_1' &= -\frac{4}{10}x_1 + \frac{1}{10}x_2\\ x_2' &= \frac{4}{10}x_1 - \frac{4}{10}x_2 \end{aligned}\right\} \quad \text{dvs.} \quad x' = \tfrac{1}{10}\begin{bmatrix}-4 & 1\\4 & -4\end{bmatrix}x$$

$$A = \begin{bmatrix}-4 & 1\\4 & -4\end{bmatrix}, \quad \text{gir} \quad \lambda^2 + 8\lambda + 12 = 0 \Rightarrow \lambda = -6, -2.$$

$$\lambda = -6: \begin{bmatrix}2 & 1\\4 & 2\end{bmatrix} \sim \begin{bmatrix}2 & 1\\0 & 0\end{bmatrix}, \quad v_1 = \begin{bmatrix}1\\-2\end{bmatrix}$$

$$\lambda = -2: \begin{bmatrix}-2 & 1\\4 & -2\end{bmatrix} \sim \begin{bmatrix}2 & -1\\0 & 0\end{bmatrix}, \quad v_2 = \begin{bmatrix}1\\2\end{bmatrix}$$

$$x(t) = c_1 e^{-\frac{3}{5}t}\begin{bmatrix}1\\-2\end{bmatrix} + c_2 e^{-\frac{1}{5}t}\begin{bmatrix}1\\2\end{bmatrix}$$

$$\begin{bmatrix} 300 \\ 200 \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ -2 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ 2 \end{bmatrix} \Rightarrow c_1 = 100,\ c_2 = 200.$$

$x_1 = 100e^{-\frac{3}{5}t} + 200e^{-\frac{1}{5}t}$

$\underline{x_2 = -200e^{-\frac{3}{5}t} + 400e^{-\frac{1}{5}t}}$