Norges teknisk–
naturvitenskapelige universitet
Institutt for matematiske fag

EKSAMEN I FAG SIF5010 MATEMATIKK 3

Lørdag 22. mai 1999

Tid: 0900-1400

Løsningsforslag

Oppgave 1

a) $z^3 = 8e^{\pi i}$ gir $z_k = 2e^{\frac{\pi+2k\pi}{3}i}$, $k = 0, 1, 2$, dvs. $z_0 = 2e^{\frac{\pi}{3}i} = 1 + \sqrt{3}i$, $z_1 = 2e^{\pi i} = -2$, $z_2 = 2e^{\frac{5\pi}{3}i} = 1 - \sqrt{3}i$.

b) $r^4 - r^3 + 8r - 8 = (r-1)(r^3+8) = 0$ gir $r = 1, -2, 1 \pm \sqrt{3}i$. Partikulær løsning $y = Axe^x$ gir $A = \frac{1}{9}$, så

$$y = c_1 e^x + c_2 e^{-2x} + c_3 e^x \cos\sqrt{3}x + c_4 e^x \sin\sqrt{3}x + \frac{1}{9}xe^x.$$

c) La $y_1 = \cos 2x$, $y_2 = \sin 2x$ med $W = 2$. Partikulær løsning er $y = u_1 y_1 + u_2 y_2$ der

$$\begin{aligned} u_1' y_1 + u_2' y_2 &= 0 \\ u_1' y_1' + u_2' y_2' &= \frac{4}{\cos 2x}, \end{aligned}$$

dvs.

$$u_1' = \frac{\begin{vmatrix} 0 & \sin 2x \\ \frac{4}{\cos 2x} & 2\cos 2x \end{vmatrix}}{W} = \frac{-2\sin 2x}{\cos 2x}, \qquad u_1 = \ln(\cos 2x)$$

$$u_2' = \frac{\begin{vmatrix} \cos 2x & 0 \\ -2\sin 2x & \frac{4}{\cos 2x} \end{vmatrix}}{W} = 2, \qquad u_2 = 2x.$$

Så

$$y = c_1 \cos 2x + c_2 \sin 2x + \cos 2x \ln(\cos 2x) + 2x \sin 2x.$$

Oppgave 2 $y_{n+1} = y_n + hf(x_n, y_n)$; $x_0 = \frac{3}{2}$, $y_0 = 1$, $h = \frac{1}{2}$ gir $y_1 = 1 + \frac{1}{2}(\frac{3}{2} + 1) = \frac{9}{4}$, $y_2 = \frac{9}{4} + \frac{1}{2}(2 + \frac{3}{2}) = 4$.

Oppgave 3

a) $$\left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 5 \\ 1 & 2 & 0 & 4 & 3 \\ 1 & 2 & \alpha & 0 & 3 \\ 1 & 2 & 0 & 5 & \beta \end{array}\right] \sim \left[\begin{array}{cccc|c} 1 & 0 & 2 & 4 & 5 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & \alpha & -4 & 0 \\ 0 & 0 & 0 & 1 & \beta - 3 \end{array}\right], ? = 0 \text{ og } \beta - 3.$$

$$A \sim I \text{ hvis } \alpha \neq 0,\ A \sim \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \text{ hvis } \alpha = 0.$$

b) nøyaktig én løsning $\Leftrightarrow \alpha \neq 0$
ingen løsning $\Leftrightarrow \alpha = 0$ og $\beta \neq 3$
uendelig mange løsninger $\Leftrightarrow \alpha = 0$ og $\beta = 3$.

c) $$[A\ \ \mathbf{b}] \sim \begin{bmatrix} 1 & 0 & 2 & 0 & 5 \\ 0 & 1 & -1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \mathbf{x} = t \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 5 \\ -1 \\ 0 \\ 0 \end{bmatrix}, t \in \mathbf{R}.$$

d) $$\text{Basis for Null}(A) : \begin{bmatrix} -2 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \text{Col}(A) : \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 4 \\ 4 \\ 0 \\ 5 \end{bmatrix},$$

$$\text{Null}(A)^{\perp} = \text{Row}(A) : \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Oppgave 4

a) $$\det(A - \lambda I) = \begin{vmatrix} 2-\lambda & 1 & 2 \\ 1 & 3-\lambda & 1 \\ 2 & 1 & 2-\lambda \end{vmatrix} = -\lambda(\lambda - 2)(\lambda - 5) = 0 \text{ gir } \lambda_1 = 0,\ \lambda_2 = 2,$$

$\lambda_3 = 5$.

$$\lambda_1 = 0 : A - \lambda I \sim \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ gir } \mathbf{x}_1 = t \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, t \neq 0$$

$$\lambda_2 = 2 : A - \lambda I \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} \text{ gir } \mathbf{x}_2 = t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, t \neq 0$$

$$\lambda_3 = 5 : A - \lambda I \sim \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ gir } \mathbf{x}_3 = t \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, t \neq 0.$$

b)

$$P = \begin{bmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\ 0 & -2/\sqrt{6} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \end{bmatrix}, \qquad D = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

Oppgave 5

$$\begin{aligned} x_1' &= -\frac{\alpha+1}{100} x_1 \\ x_2' &= \frac{1}{100} x_1 - \frac{1}{50} x_2 \\ x_3' &= \frac{\alpha}{100} x_1 + \frac{1}{50} x_2 - \frac{\alpha+1}{100} x_3, \quad \text{dvs.} \end{aligned}$$

$$\mathbf{x}' = \tfrac{1}{100} \begin{bmatrix} -(\alpha+1) & 0 & 0 \\ 1 & -2 & 0 \\ \alpha & 2 & -(\alpha+1) \end{bmatrix} \mathbf{x} = \tfrac{1}{100} A\mathbf{x}.$$

a) $\alpha = 2$ gir $A = \begin{bmatrix} -3 & 0 & 0 \\ 1 & -2 & 0 \\ 2 & 2 & -3 \end{bmatrix}$ med $\lambda_1 = -3, \ \lambda_2 = -2.$

$$A - \lambda_1 I \sim \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ gir } \mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$A - \lambda_2 I \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ gir } \mathbf{v} = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Så

$$\mathbf{x} = c_1 e^{-\frac{3}{100}t} \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + c_2 e^{-\frac{3}{100}t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_3 e^{-\frac{1}{50}t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

$$\begin{bmatrix} 20 \\ 0 \\ 20 \end{bmatrix} = \mathbf{x}(0) = \begin{bmatrix} c_1 \\ -c_1 + c_3 \\ c_2 + 2c_3 \end{bmatrix} \text{ gir } c_1 = c_3 = 20 \;\; c_2 = -20,$$

så

$$\mathbf{x} = 20 e^{-\frac{3}{100}t} \begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix} + 20 e^{-\frac{1}{50}t} \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

b) $\alpha = 3$ gir $A = \begin{bmatrix} -4 & 0 & 0 \\ 1 & -2 & 0 \\ 3 & 2 & -4 \end{bmatrix}$ med $\lambda_1 = -4, \;\; \lambda_2 = -2.$

$$A - \lambda_1 I \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \text{ gir } \mathbf{v_1} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \text{ (defekt egenverdi),}$$

$$A - \lambda_2 I \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} \text{ gir } \mathbf{v_2} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Siden λ_1 er en defekt egenverdi finnes den andre løsningen fra denne ved å løse ligningen

$$(A - \lambda_1 I)\mathbf{u} = \mathbf{v}_1$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 3 & 2 & 0 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \sim \left[\begin{array}{ccc|c} 2 & 0 & 0 & 1 \\ 0 & 2 & 0 & -1/2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

som gir

$$\mathbf{u} = \begin{bmatrix} 1/2 \\ -1/4 \\ 0 \end{bmatrix}$$

Så

$$\mathbf{x} = c_1 e^{-\frac{1}{25}t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 e^{-\frac{1}{25}t} \left(\frac{t}{100} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 1/2 \\ -1/4 \\ 0 \end{bmatrix} \right) + c_3 e^{-\frac{1}{50}t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

$$\begin{bmatrix} 20 \\ 0 \\ 20 \end{bmatrix} = \mathbf{x}(0) = \begin{bmatrix} c_2/2 \\ -c_2/4 + c_3 \\ c_1 + c_3 \end{bmatrix} \text{ gir } c_1 = 10, c_2 = 40, c_3 = 10,$$

så

$$\mathbf{x} = 10 e^{-\frac{1}{25}t} \left(\frac{t}{100} \begin{bmatrix} 0 \\ 0 \\ 4 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix} \right) + 10 e^{-\frac{1}{50}t} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Oppgave 6 Med $B = [\mathbf{b}_1\ \ \mathbf{b}_2 \cdots \mathbf{b}_p]$ er $AB = [A\mathbf{b}_1\ \ A\mathbf{b}_2 \cdots A\mathbf{b}_p]$, og $AB = \mathit{0}$ gir $A\mathbf{b}_i = \mathbf{0}$, $1 \leq i \leq p$. Siden $\mathrm{Col}(B) = \mathrm{span}\{\mathbf{b}_1, \cdots, \mathbf{b}_p\}$, er da $A\mathbf{x} = \mathbf{0}$ for alle $\mathbf{x} \in \mathrm{Col}(B)$, og $\mathrm{Col}(B) \subseteq \mathrm{Null}(A)$.