Problem 1 Find all solutions of the equation $z^4 = \frac{-5+i\sqrt{3}}{2+i\sqrt{3}}$. Write your answer in Cartesian (normal) form with exact values, and draw the solutions in the complex plane.

Solution:

We begin by simplifying the given expression:

$$\begin{aligned}\frac{-5+i\sqrt{3}}{2+i\sqrt{3}} &= \frac{-5+i\sqrt{3}}{2+i\sqrt{3}} \bullet \frac{2-i\sqrt{3}}{2-i\sqrt{3}} \\ &= \frac{(-5+i\sqrt{3})(2-i\sqrt{3})}{(2+i\sqrt{3})(2-i\sqrt{3})} \\ &= \frac{-10+3+i\sqrt{3}(5+2)}{2^2+3} \\ &= \frac{-7+i7\sqrt{3}}{7} \\ &= -1+i\sqrt{3}\end{aligned}$$

To find the 4th roots of this, we write it in polar form.

$$-1+i\sqrt{3} = 2e^{2\pi i/3}.$$

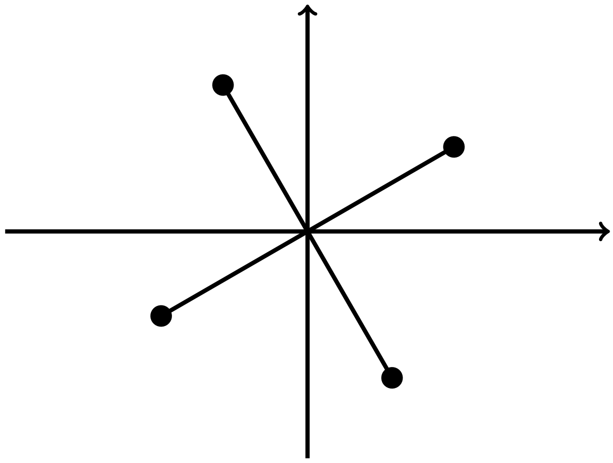
Hence we have the following polar forms for z:

$$\sqrt[4]{2}e^{\pi i/6}, \quad \sqrt[4]{2}e^{2\pi i/3}, \quad \sqrt[4]{2}e^{7\pi i/6}, \quad \sqrt[4]{2}e^{5\pi i/3}$$

These have the following Cartesian forms (using the fact that $\sqrt[4]{2}/2 = 1/\sqrt[4]{8}$):

$$1/\sqrt[4]{8}(\sqrt{3}+i), \quad 1/\sqrt[4]{8}(-1+i\sqrt{3}), \quad 1/\sqrt[4]{8}(-\sqrt{3}-i), \quad 1/\sqrt[4]{8}(1-i\sqrt{3})$$

And on the complex plane, these are as follows:



Problem 2

a) Find the general solution of $y'' + y' - 2y = 0$.

Solution:

This is a homogeneous second order linear ODE with constant coefficients so it will have solutions of the form e^{kt}. To find out k, we substitute in to get:

$$k^2e^{kt} + ke^{kt} - 2e^{kt} = 0$$

Dividing by e^{kt} produces:

$$k^2 + k - 2 = 0$$

which has solutions $k = -2$ and $k = 1$. Hence the general solution is of the form:

$$Ae^{-2t} + Be^{t}.$$

b) Find the solution of $y'' + y' - 2y = 10\cos t + 1 - 2t^2$ with initial conditions $y(0) = 11$, $y'(0) = 3$.

Solution:

Since we already have the general solution of the homogeneous equation, we look for a particular solution.

The right-hand side is a sum, so we split it into its pieces. The first is $10\cos t$ so we guess a solution of the form $y(t) = A\cos t + B\sin t$. Substituting in, we get:

$$\begin{aligned} y''(t) + y'(t) - 2y(t) &= -A\cos t - B\sin t - A\sin t + B\cos t - 2A\cos t - 2B\sin t \\ &= (-3A + B)\cos t + (-3B - A)\sin t \end{aligned}$$

To get this equal to $10\cos t$ we need $-3A + B = 10$ and $-3B - A = 0$. Thus $B = 1$ and $A = -3$.

Now we consider the term $1 - 2t^2$. For this, we guess a solution of the form $y(t) = A + Bt + Ct^2$. Substituting in, we get:

$$\begin{aligned} y''(t) + y'(t) - 2y(t) &= 2C + 2Ct + B - 2A - 2Bt - 2Ct^2 \\ &= (2C + B - 2A) + (2C - 2B)t + (-2C)t^2 \end{aligned}$$

To get this equal to $1 - 2t^2$ we must have $C = 1$, whence also $B = 1$, and thus $A = 1$.

So our particular solution is:

$$-3\cos t + \sin t + 1 + t + t^2$$

Our general solution is thus:

$$y(t) = -3\cos t + \sin t + 1 + t + t^2 + Ae^{-2t} + Be^t$$

At $t = 0$, we get $y(0) = -3 + 1 + A + B$ and $y'(0) = 1 + 1 - 2A + B$. Thus we must find A and B such that:

$$\begin{aligned} -2 + A + B &= 11 \\ 2 - 2A + B &= 3 \end{aligned}$$

So $-3A = -12$, whence $A = 4$ and $B = 9$. Thus the solution is:

$$-3\cos t + \sin t + 1 + t + t^2 + 4e^{-2t} + 9e^t.$$

Problem 3 Let

$$A = \begin{bmatrix} 1 & -2 & 2 & -4 & 3 \\ -2 & 4 & 0 & -4 & -5 \\ 4 & -8 & 3 & -1 & 7 \\ 3 & -6 & 1 & 3 & 0 \end{bmatrix}$$

a) Find a basis for the column space, Col(A), and a basis for the null space, Null(A), of the matrix A.

Solution:

We row reduce A as follows. We do a full row reduction to get the simplest form for the rows since we will be working with them in the next part.

$$\begin{bmatrix} 1 & -2 & 2 & -4 & 3 \\ -2 & 4 & 0 & -4 & -5 \\ 4 & -8 & 3 & -1 & 7 \\ 3 & -6 & 1 & 3 & 0 \end{bmatrix} \xrightarrow[\substack{\rho_3-4\rho_1 \\ \rho_4-3\rho_1}]{\rho_2+2\rho_1} \begin{bmatrix} 1 & -2 & 2 & -4 & 3 \\ 0 & 0 & 4 & -12 & 1 \\ 0 & 0 & -5 & 15 & -5 \\ 0 & 0 & -5 & 15 & -9 \end{bmatrix}$$

$$\xrightarrow{\rho_2\leftrightarrow\rho_3} \begin{bmatrix} 1 & -2 & 2 & -4 & 3 \\ 0 & 0 & -5 & 15 & -5 \\ 0 & 0 & 4 & -12 & 1 \\ 0 & 0 & -5 & 15 & -9 \end{bmatrix}$$

$$\xrightarrow{-\rho_2/5} \begin{bmatrix} 1 & -2 & 2 & -4 & 3 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 4 & -12 & 1 \\ 0 & 0 & -5 & 15 & -9 \end{bmatrix}$$

$$\xrightarrow[\substack{\rho_4+5\rho_2 \\ \rho_1-2\rho_2}]{\rho_3-4\rho_2} \begin{bmatrix} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & -3 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow{-\rho_3/3} \begin{bmatrix} 1 & -2 & 0 & 2 & 1 \\ 0 & 0 & 1 & -3 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -4 \end{bmatrix}$$

$$\xrightarrow[\substack{\rho_1-\rho_3 \\ \rho_2-\rho_3}]{\rho_4+4\rho_3} \begin{bmatrix} 1 & -2 & 0 & 2 & 0 \\ 0 & 0 & 1 & -3 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

From this, we see that columns 1, 3, and 5 are pivot columns and thus a basis of Col(A) comes from the same columns in the original matrix:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 4 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ -5 \\ 7 \\ 0 \end{bmatrix} \right\}$$

Columns 2 and 4 are free and from them we get a basis of the null space by setting each free variable to 1 in turn.

$$\left\{\begin{bmatrix}2\\1\\0\\0\\0\end{bmatrix}, \begin{bmatrix}-2\\0\\3\\1\\0\end{bmatrix}\right\}$$

b) Find an orthogonal basis for the row space of the matrix A.

Solution:

The row reduced form shows that the following is a basis of Row(A):

$$\left\{\begin{bmatrix}1\\-2\\0\\2\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\-3\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\\0\\1\end{bmatrix}\right\}$$

so we apply Gram–Schmidt to this family. Thus we start with $\vec{w}_1 = \begin{bmatrix}1\\-2\\0\\2\\0\end{bmatrix}$. We compute $\vec{w}_1 \bullet \vec{w}_1 = 9$. The next step is to replace $\begin{bmatrix}0\\0\\1\\-3\\0\end{bmatrix}$ by:

$$\begin{bmatrix}0\\0\\1\\-3\\0\end{bmatrix} - \left(\frac{1}{9}\begin{bmatrix}0\\0\\1\\-3\\0\end{bmatrix} \bullet \begin{bmatrix}1\\-2\\0\\2\\0\end{bmatrix}\right)\begin{bmatrix}1\\-2\\0\\2\\0\end{bmatrix} = \begin{bmatrix}2/3\\-4/3\\1\\-5/3\\0\end{bmatrix}$$

We can multiply by 3 since we only want an orthogonal basis. Thus we take $\vec{w}_2 = \begin{bmatrix}2\\-4\\3\\-5\\0\end{bmatrix}$.

The last step is to consider $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. But this is already orthogonal to $\vec{w}_1$ and $\vec{w}_2$ so there is nothing to do here. Thus we have basis

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -4 \\ 3 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

This basis is not unique. Different starting points will produce different bases.

c) Let T be the linear transformation with matrix A. Is T one-to-one? Is it onto? Justify your answers.

Solution:

As A has a non-trivial null space, there is a non-zero vector $\vec{w}$ such that $A\vec{w} = \vec{0}$. Hence T cannot be one-to-one.

The row reduced form of A has a row of zeros. Hence there is a vector $\vec{b}$ such that $A\vec{x} = \vec{b}$ does not have a solution. Thus T cannot be onto.

Problem 4 Let P_2 be the space of all polynomials of degree less than or equal to two. What is the dimension of P_2?

Let $p_1(t) = t$, $p_2(t) = t(t-1)$, and $p_3(t) = (t-1)(t-2)$. Is $\{p_1, p_2, p_3\}$ a basis for P_2? Justify your answer.

Solution:

Every polynomial of degree at most two is of the form:

$$a_0 + a_1 t + a_2 t^2 = a_0 1 + a_1 t + a_2 t^2$$

for some $a_0, a_1, a_2 \in \mathbb{R}$. This is the zero polynomial if and only if $a_0 = a_1 = a_2 = 0$. Hence we have a basis $\{1, t, t^2\}$ for P_2 and thus $\dim P_2 = 3$.

In terms of the basis $\mathscr{B} := \{1, t, t^2\}$, the given set of polynomials are:

$$\left[p_1(t)\right]_{\mathscr{B}} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

$$\left[p_2(t)\right]_{\mathscr{B}} = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}$$

$$\left[p_3(t)\right]_{\mathscr{B}} = \begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix}$$

As there are three of them, and $\dim P_2 = 3$, they will form a basis if and only if they are linearly independent. We test this by row reducing the matrix of the corresponding column vectors.

$$\begin{bmatrix} 0 & 0 & 2 \\ 1 & -1 & -3 \\ 0 & 1 & 1 \end{bmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_2} \begin{bmatrix} 1 & -1 & -3 \\ 0 & 0 & 2 \\ 0 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\rho_2 \leftrightarrow \rho_3} \begin{bmatrix} 1 & -1 & -3 \\ 0 & 1 & 1 \\ 0 & 0 & 2 \end{bmatrix}$$

This has full rank, so its columns are linearly independent. Hence $\{p_1, p_2, p_3\}$ is a basis for P_3.

Problem 5 In Sommerby the rental company has three locations for renting out boats: Market, Island, and Camping. The pattern of returns to the rental locations is the following: for boats rented at Market, one-quarter is returned to Market, one-half to Island, and one-quarter to Camping; half of the boats rented at Island are returned to Market and half to Camping; for boats rented at Camping one-sixth are returned to Market, one-half to Island, and one-third to Camping. Find the stochastic matrix P that describes how the distribution of boats changes. Find the steady-state vector for P.

Solution:

The stochastic matrix is formed by listing the proportions that go from one place to another. We list the places in the order: Market, Island, Camping. The i, j–entry of the matrix P is then the proportion that goes from the jth location to the ith location.

Thus the matrix is:

$$P := \begin{bmatrix} 1/4 & 1/2 & 1/6 \\ 1/2 & 0 & 1/2 \\ 1/4 & 1/2 & 1/3 \end{bmatrix}$$

The steady-state vector is found by first looking for eigenvectors of P with eigenvalue 1. That is, we look for solutions of $P\vec{x} = \vec{x}$ or equivalently for the null space of $P - I$. Thus we row reduce $P - I$ as follows:

$$\begin{bmatrix} -3/4 & 1/2 & 1/6 \\ 1/2 & -1 & 1/2 \\ 1/4 & 1/2 & -2/3 \end{bmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_3} \begin{bmatrix} 1/4 & 1/2 & -2/3 \\ 1/2 & -1 & 1/2 \\ -3/4 & 1/2 & 1/6 \end{bmatrix}$$
$$\underset{\rho_3+3\rho_1}{\xrightarrow{\rho_2-2\rho_1}} \begin{bmatrix} 1/4 & 1/2 & -2/3 \\ 0 & -2 & 11/6 \\ 0 & 2 & -11/6 \end{bmatrix}$$

From this, we can read off that the null space is spanned by the vector:

$$\begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$$

This is not a state vector as it doesn't sum to 1. Dividing by its sum, we get that the steady-state vector is:

$$\frac{1}{33}\begin{bmatrix} 10 \\ 11 \\ 12 \end{bmatrix}$$

Problem 6 Find the solution of the system

$$x_1' = x_1 + 2x_2$$
$$x_2' = 3x_1 + 2x_2$$

that satisfies the initial conditions $x_1(0) = 1$ and $x_2(0) = 1$.

Solution:

We start by writing this in matrix form as:

$$\vec{x}'(t) = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix} \vec{x}(t)$$

Let $A = \begin{bmatrix} 1 & 2 \\ 3 & 2 \end{bmatrix}$. We solve this problem by looking for eigenvectors and eigenvalues of A.

To find the eigenvalues, we consider the characteristic polynomial of A, $\det(A - \lambda I)$:

$$\det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 2 \\ 3 & 2-\lambda \end{vmatrix} = (1-\lambda)(2-\lambda) - 6 = \lambda^2 - 3\lambda - 4$$

This has roots 4 and -1. These are thus the eigenvalues of A.

To find the eigenvectors, we look at the null spaces of $A - 4I$ and $A + I$.

$$\begin{bmatrix} -3 & 2 \\ 3 & -2 \end{bmatrix} \mapsto \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix} \mapsto \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Thus eigenvalues are $\begin{bmatrix} 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

The general solution is thus:

$$\vec{x}(t) = Ae^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} + Be^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

We need to fit this to the initial conditions. At $t = 0$ we have:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = \vec{x}(0) = A \begin{bmatrix} 2 \\ 3 \end{bmatrix} + B \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

This has solution $A = 2/5$, $B = -1/5$. Thus the solution of the ODE is:

$$\vec{x}(t) = \frac{2}{5} e^{4t} \begin{bmatrix} 2 \\ 3 \end{bmatrix} - \frac{1}{5} e^{-t} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Written out in non-vector form, this is:

$$x_1(t) = 4/5 e^{4t} + 1/5 e^{-t}$$
$$x_2(t) = 6/5 e^{4t} - 1/5 e^{-t}$$

Problem 7 Find the least squares line $y = mx + c$ that best fits the data points:

$$\{(0,3),(1,3),(2,6),(3,-3),(4,1),(5,-1)\}$$

Solution:

We write this in matrix form as follows:

$$\begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \\ 5 & 1 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ 6 \\ -3 \\ 1 \\ -1 \end{bmatrix}$$

This does not have a solution so we look for a least-squares solution.

Setting A to be the matrix in the above and $\vec{b}$ the vector on the right-hand side, we examine the linear system $A^\top A\vec{x} = A^\top \vec{b}$. This is:

$$\begin{bmatrix} 55 & 15 \\ 15 & 6 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 9 \end{bmatrix}$$

To solve this, we row reduce the augmented matrix. One possible row reduction is as follows.

$$\begin{bmatrix} 55 & 15 & 5 \\ 15 & 6 & 9 \end{bmatrix} \xrightarrow[1/3\rho_2]{1/5\rho_1} \begin{bmatrix} 11 & 3 & 1 \\ 5 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{\rho_1 - 2\rho_2} \begin{bmatrix} 1 & -1 & -5 \\ 5 & 2 & 3 \end{bmatrix}$$

$$\xrightarrow{\rho_2 - 5\rho_1} \begin{bmatrix} 1 & -1 & -5 \\ 0 & 7 & 28 \end{bmatrix}$$

$$\xrightarrow{1/7\rho_2} \begin{bmatrix} 1 & -1 & -5 \\ 0 & 1 & 4 \end{bmatrix}$$

$$\xrightarrow{\rho_1 + \rho_2} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 4 \end{bmatrix}$$

Thus the solution is $m = -1$, $c = 4$ and so the least-squares line is $y = -x + 4$.