Problem 1 :

Assume $z^3 = 1$, we can write this as

$$z^3 = e^{i 2\pi k} , k = 0, 1, 2, \dots$$

Hence

$$z = e^{i \frac{2\pi k}{3}} \quad k = 0, 1, 2, \dots$$

We get 3 different solutions :

$$z_1 = e^{i0} = 1$$

$$z_2 = e^{i\frac{2\pi}{3}} = \cos\frac{2\pi}{3} + i \sin\frac{2\pi}{3} = \frac{1}{2}(-1 + i\sqrt{3})$$

$$z_3 = e^{i\frac{4\pi}{3}} = \cos\frac{4\pi}{3} + i \sin\frac{4\pi}{3} = -\frac{1}{2}(1 + i\sqrt{3})$$

To solve the equation $z^3 = \frac{-3+i}{\sqrt{2}(2+i)}$ we first need to rewrite the right hand side

$$z^3 = \frac{(-3+i)(2-i)}{\sqrt{2}(2+i)(2-i)} = \frac{-6+2i+3i+1}{\sqrt{2}\;(4+1)}$$

$$= -\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i = e^{i\frac{3\pi}{4} + 2\pi k i} , i = 0, 1, 2, \dots$$

Hence $z = e^{i\frac{\pi}{4} + \frac{2\pi k}{3} i}$

$$z = e^{i\frac{\pi}{4}} e^{\frac{2\pi k}{3} i} \text{ or } = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) e^{\frac{2\pi k i}{3}}, \quad k = 0, 1, \ldots$$

We use the first part to see that:

$$z_1 = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \cdot 1 = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)$$

$$z_2 = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \frac{1}{2}\left(-1 + i\sqrt{3}\right)$$

or

$$z_2 = \frac{1}{2\sqrt{2}} \left(-1 - \sqrt{3} + i\left(\sqrt{3} - 1\right)\right)$$

and

$$z_3 = \left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) \frac{1}{2}\left(-1 - i\sqrt{3}\right)$$

or

$$z_3 = \frac{1}{2\sqrt{2}} \left(-1 + \sqrt{3} - i\left(\sqrt{3} + 1\right)\right)$$

Problem 2

$$y'' - 2y' + y = \frac{e^x}{x}$$

First we solve $y'' - 2y' + y = 0$

The characteristic equation is

$$\lambda^2 - 2\lambda + 1 = 0$$

We get a double root $\lambda = 1$
so

$$y_1 = e^x \quad , \quad y_2 = x e^x$$

are two linearly independent solutions
and

$$y_h = C_1 e^x + C_2 x e^x$$

To find a particular solution to the inhomogeneous equation we need to use variation of parameter.
We look for

$$y_p = v_1 y_1 + v_2 y_2 = v_1 e^x + v_2 x e^x$$

where

$$v_1' y_1 + v_2' y_2 = 0$$

and

$$v_1' y_1' + v_2' y_2' = \frac{e^x}{x}$$

We find that

$$v_1 = \int dx = x$$

and

$$v_2 = \int \frac{1}{x}\, dx = \ln x$$

a particular solution would be

$$y_p = x e^x + x \ln x \, e^x$$

The solutions are

$$y = y_h + y_p = C_1 e^x + C_2 x e^x + x e^x + x \ln x \, e^x$$

notice that we can absorb $x e^x$ in to $C_2 x e^x$

$$y = C_1 e^x + C_2 x e^x + x \ln x \, e^x$$

Now

$$y(1) = C_1 e + C_2 e = 0$$

and

$$y'(1) = C_1 e + C_2 e + C_2 e + e = 0$$

Hence $c_1 = 1$ and $c_2 = 1$
and our solution is

$$y = e^x - x e^x + x \ln x \, e^x$$

PROBLEM 3

$$y'' + 2c y' + 4y = 0$$

have characteristic equation

$$\lambda^2 + 2c\lambda + 4 = 0$$

so the solutions are
on the form

$$\lambda = \frac{-2c \pm \sqrt{4c^2 - 16}}{2}$$

1. We get complex solutions if
$4c^2 - 16 < 0$

So it is underdamped if

$4c^2 < 16$ or $c^2 < 4$
Hence
$0 < c < 2$

(2) We only get one root if

$$4c^2 - 16 = 0 \quad \text{or} \quad c = 2$$

Critically damped if $c = 2$

(3) We get two negativ real roots if
$$4c^2 - 16 > 0$$

Overdamped if $c > 2$.

To find the steady-state solution to
$$y'' + 2y' + 4y = \cos t$$

We need to find a particular solution.

This time we will try

$$y_p = A\cos t + B\sin t$$
then
$$y_p' = -A\sin t + B\cos t$$

$$y_p'' = -A\cos t - B\sin t$$

When we plug this in to the equation we get.

$$(3A+2B)\cos t + (3B-2A)\sin t = \cos t$$

so

$$3A+2B = 1$$

and

$$3B-2A = 0$$

and

$$A = \frac{3}{13} \quad \text{while} \quad B = \frac{2}{13}$$

so

$$y_s = \frac{1}{13}(3\cos t + 2\sin t)$$

PROBLEM 3:

Observe that this problem have many correct answers.

$$T : \mathbb{R}^4 \to \mathbb{R}$$

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = 2x + 2y - z + w$$

The null space for T

$$\text{Null } T = \left\{ \begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in \mathbb{R}^4 : 2x + 2y - z + w = 0 \right\}$$

We need to solve the equation.

$$2x + 2y - z + w = 0$$

Notice we have one equation with four unknown so we have 3 degrees of freedom.

1. We may choose $z = w = 0$ and $x = 1$ this implies that $y = -1$

$$\vec{w}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$$

2. Next we may choose $x = y = 0$ and $z = 1$ then $w = 1$

$$\vec{w}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

3. We can choose $x = 0$, $y = 1$, $w = 0$ then $z = 2$

$$\vec{w}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix}$$

If we try to find a, b and c such that

$$a\vec{w}_1 + b\vec{w}_2 + c\vec{w}_3 = 0$$

We see that $a = b = c = 0$

Hence $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is linearly independent

The dimension of Null T is 3

hence $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is a basis for this space

Next we need an orthogonal basis

We see that

$\vec{w}_1 \cdot \vec{w}_2 = 0$ so we only need to use Gram Schmidt to change $\vec{w}_3$

Let $\vec{v}_1 = \vec{w}_1$ and $\vec{v}_2 = \vec{w}_2$

and

$$\vec{v}_3 = \vec{w}_3 - \frac{\vec{w}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\vec{v}_1 - \frac{\vec{w}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\vec{v}_2$$

$$\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} - \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ -1 \end{bmatrix}$$

To get an orthonormal basis we need to replace $\vec{v}_1$, $\vec{v}_2$ and $\vec{v}_3$ by

$$\frac{\vec{v}_1}{\|\vec{v}_1\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix} \qquad \frac{\vec{v}_2}{\|\vec{v}_2\|} = \frac{1}{\sqrt{2}}\begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

$$\frac{\vec{v}_3}{\|\vec{v}_3\|} = \frac{2}{\sqrt{10}}\begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ 1 \\ -1 \end{bmatrix} = \frac{1}{\sqrt{10}}\begin{bmatrix} 1 \\ 1 \\ 2 \\ -2 \end{bmatrix}.$$

PROBLEM 5

First we compute $\det A$

$$\det A = \det \begin{bmatrix} 1 & 0 & 0 & 0 \\ -t & 1 & 0 & 1 \\ 0 & -t & 1 & 0 \\ 0 & 0 & -t & -1 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 0 & 1 \\ -t & 1 & 0 \\ 0 & -t & -1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 \\ -t & -1 \end{bmatrix} + \det \begin{bmatrix} -t & 1 \\ 0 & -t \end{bmatrix}$$

$$= -1 + t^2$$

So when $t^2 - 1 \neq 0$ or $t \neq \pm 1$, then

$\det A \neq 0$ and the equation

$$Ax = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}$$

have one and only one

solution.

If $t = -1$ then $A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{bmatrix}$

To solve the equation we row reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{II - I} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{III - IV} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right] \xrightarrow{III - II} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 \end{array}\right]$$

From this we learn that if

$\vec{X} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is a solution we

need

$$x_1 = 0$$

$$x_2 + x_4 = 1$$

$$x_3 - x_4 = 0$$

Choose x_4 free ~~then~~ then

$x_3 = x_4$ and $x_2 = 1 - x_4$ so

$$\vec{X} = \begin{bmatrix} 0 \\ 1-x_4 \\ x_4 \\ x_4 \end{bmatrix}$$

so there are infinitely many solution.

If $t = 1$ then

$$A = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & -1 \end{bmatrix}$$

To solve the equation

we row reduce

$$\left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 0 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow[\overline{III}+\overline{IV}]{\overline{II}+\overline{I}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right]$$

$$\xrightarrow{\overline{II}+\overline{III}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \\ 0 & -1 & 0 & -1 & 1 \\ 0 & 0 & -1 & -1 & 0 \end{array}\right]$$

From the final row $\overline{II}$ we see there is no solution.

PROBLEM 6:

We are looking for what values of a our matrix have two linearly independent eigenvectors in $\mathbb{R}^2$

We find the characteristic equation

$$\det\left(\begin{bmatrix} 0 & a \\ 1 & 0 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right)$$

$= \lambda^2 - a = 0$ so

$\lambda^2 = a$

Case I $a > 0$ then $\lambda_1 = \sqrt{a}$ and $\lambda_2 = -\sqrt{a}$

and $\lambda_1 \neq \lambda_2$ are real

The corresponding eigenvectors will be real and linearly independent.

Case 2 : $a = 0$ so $\lambda_1 = \lambda_2 = 0$

Eigenvectors need to satisfy

$$\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

so $x = 0$ and all the eigenvectors will be on the form.

$$y \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

We do NOT have two linearly independent eigenvectors.

<u>Case 3:</u>

$a < 0$

then λ_1 and λ_2 are imaginary.

$$\lambda_1 = i\sqrt{-a} \quad \text{and} \quad \lambda_2 = -i\sqrt{-a}$$

Eigenvectors for λ_1 will have to satisfy.

$$\begin{bmatrix} -i\sqrt{-a} & a \\ 1 & -i\sqrt{-a} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

they will be real multiples of

$$\begin{bmatrix} i\sqrt{-a} \\ 1 \end{bmatrix}$$ hence not in $\mathbb{R}^2$

for λ_2 they will be real multiples of

$$\begin{bmatrix} -i\sqrt{-a} \\ 1 \end{bmatrix}.$$ hence ~~also~~ also not in $\mathbb{R}^2$

Answer to the question is $a > 0$.

PROBLEM 7 :

The rate of change of salt contained in tank T_1 is

$$x_1'(t) = -\frac{1}{100}x_1(t) + \frac{1}{100}x_2(t)$$

and the rate of change in tank T_2 is

$$x_2'(t) = \frac{1}{100}x_1(t) - \frac{1}{100}x_2(t)$$

or.

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \end{bmatrix} = \begin{bmatrix} -\frac{1}{100}x_1(t) + \frac{1}{100}x_2(t) \\ \frac{1}{100}x_1(t) - \frac{1}{100}x_2(t) \end{bmatrix}$$

$$= \begin{bmatrix} -\frac{1}{100} & \frac{1}{100} \\ \frac{1}{100} & -\frac{1}{100} \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$= \frac{1}{100}\begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

$$= A \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}$$

In order to solve the system we need to find the eigenvalues and eigenvectors of A

We start by solving.

$$\det\left(\begin{bmatrix} -\frac{1}{100} & \frac{1}{100} \\ \frac{1}{100} & -\frac{1}{100} \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)$$

$$= \left(-\frac{1}{100} - \lambda\right)^2 - \left(\frac{1}{100}\right)^2 = 0$$

or.

$$\left(\lambda + \frac{1}{100}\right)^2 = \left(\frac{1}{100}\right)^2$$

We get

$$\lambda_1 + \frac{1}{100} = \frac{1}{100} \quad \text{or} \quad \lambda_1 = 0$$

and

$$\lambda_2 + \frac{1}{100} = -\frac{1}{100} \quad \text{or} \quad \lambda_2 = -\frac{1}{50}$$

Next we need to find the belonging eigenvectors.

$\underline{\lambda_1 = 0}$

$$\begin{bmatrix} -\frac{1}{100} & \frac{1}{100} \\ \frac{1}{100} & -\frac{1}{100} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Which gives

$$-u_1 + u_2 = 0$$

Let $u_1 = 0$, then $u_2 = 1$ and

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$\lambda_2 = -\frac{1}{50}$

$$\begin{bmatrix} \frac{1}{100} & \frac{1}{100} \\ \frac{1}{100} & \frac{1}{100} \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

or.

$$u_1 + u_2 = 0$$

Let $u_1 = 1$, then $u_2 = -1$

$$\vec{v}_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

The general solution will be on the form

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{0t} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-\frac{1}{50}t}$$

$$= c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} e^{-\frac{1}{50}t}$$

Now $x_1(0) = 100\,g$ while $x_2(0) = 0\,g$

so

$$c_1 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 100 \\ 0 \end{bmatrix}$$

or

$$c_1 + c_2 = 100$$
$$c_1 - c_2 = 0$$

Hence $c_1 = c_2 = 50$

Our solution will be on the form.

$$x_1(t) = 50(1 + e^{-\frac{t}{50}})$$

$$x_2(t) = 50(1 - e^{-\frac{t}{50}})$$

When is $x_2(t) = 25$

We set

$$25 = 50(1 - e^{-\frac{t}{50}})$$

or.

$$1 - e^{-\frac{t}{50}} = \frac{1}{2}$$

or

$$e^{-\frac{t}{50}} = \frac{1}{2}$$

Hence $-\frac{t}{50} = \log \frac{1}{2}$

or.

$$t = 50 \log 2$$