NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for TMA4115 Matematikk 3

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Give reasons for all answers, ensuring that it is clear how the answer has been reached. Each of the parts has the same weight.

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Problem 1 Find all solutions of $\operatorname{Im} z = \frac{\sqrt{2}}{2}|z|$ and draw them on the [illegible] plane.

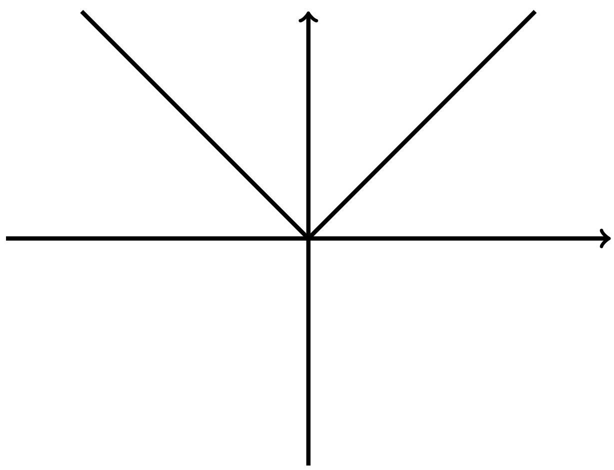
Solution:

We can solve this using either polar form or cartesian form.

If using polar form, we write $z = re^{i\theta}$ and recall that $\operatorname{Im} z = r \sin\theta$ and $|z| = r$. Thus the equation is $r\sin\theta = \frac{\sqrt{2}}{2}r$. We get a solution if $r = 0$, and if $r \neq 0$ then we can cancel it to leave $\sin\theta = \frac{\sqrt{2}}{2}$. The solutions of this are $\theta = \pi/4$ and $\theta = 3\pi/4$. In polar form, then, the solutions are $z = re^{i\theta}$ with either $r = 0$ or $\theta = \pi/4$ or $\theta = 3\pi/4$. Converting these into standard (or cartesian) form, we have $z = r\frac{\sqrt{2}}{2}(1+i)$ and $z = r\frac{\sqrt{2}}{2}(-1+i)$ for $r \geq 0$. Equivalently, $z = x + i|x|$ for $x \in \mathbb{R}$.

In cartesian form, we write $z = x + iy$ and substitute in. This yields $y = \frac{\sqrt{2}}{2}\sqrt{x^2+y^2}$. We can square both sides to get $y^2 = \frac{1}{2}(x^2+y^2)$ which rearranges to $y^2 = x^2$. The solutions of this are $y = \pm x$ with $x \in \mathbb{R}$. However, when we squared both sides we lost some information. The equation $y = \frac{\sqrt{2}}{2}\sqrt{x^2+y^2}$ forces y to be positive. Thus the solutions to the original are $y = |x|$ with $x \in \mathbb{R}$. Hence $z = x + i|x|$ for $x \in \mathbb{R}$.

We can draw these on the complex plane as follows.



Problem 2 Find a particular solution of the differential equation:

$$y''(t) + 3y'(t) + 2y(t) = 2e^{-t}.$$

Solution:

The right-hand side is a simple exponential so we try the method of undetermined coefficients. Substituting in $y(t) = e^{-t}$, we see that:

$$\begin{aligned} y''(t) + 3y'(t) + 2y(t) &= e^{-t} - 3e^{-t} + 2e^{-t} \\ &= (1 - 3 + 2)e^{-t} \\ &= 0. \end{aligned}$$

That is, e^{-t} is a solution of the homogeneous equation.

The rule for this, then, is to try multiplying by t; that is, to try $y(t) = te^{-t}$. Then $y'(t) = -te^{-t} + e^{-t}$ and $y''(t) = te^{-t} - 2e^{-t}$. Substituting in we obtain:

$$\begin{aligned} y''(t) + 3y'(t) + 2y(t) &= te^{-t} - 2e^{-t} - 3te^{-t} + 3e^{-t} + 2te^{-t} \\ &= (1 - 3 + 2)te^{-t} + (-2 + 3)e^{-t} \\ &= e^{-t}. \end{aligned}$$

Thus to get $2e^{-t}$ we need to start with $2te^{-t}$. Hence a particular solution is $2te^{-t}$.

Problem 3 Given that $y_1(x) = x^{-1}$ and $y_2(x) = x^{-2}$ are two linearly independent solutions of the differential equation

$$y''(x) + 4x^{-1}y'(x) + 2x^{-2}y(x) = 0, \qquad x > 0,$$

find the general solution of the differential equation:

$$y''(x) + 4x^{-1}y'(x) + 2x^{-2}y(x) = x^{-3}, \qquad x > 0.$$

Solution:

As we are given two linearly independent solutions of the homogeneous equation, this looks suitable for variation of parameters. Thus we look for a solution of the form $y(x) = u_1(x)y_1(x) + u_2(x)y_2(x)$. The functions u_1 and u_2 are assumed to satisfy:

$$\begin{aligned} u_1'y_1 + u_2'y_2 &= 0, \\ u_1'y_1' + u_2'y_2' &= r. \end{aligned}$$

The ODE is already in standard form so $r(x) = x^{-3}$. Substituting in, we obtain:

$$u_1'x^{-1} + u_2'x^{-2} = 0,$$
$$u_1'(-x^{-2}) + u_2'(-2x^{-3}) = x^{-3}.$$

The first equation simplifies to $u_2' = -xu_1'$. The second equation simplifies to $xu_1' + 2u_2' = -1$. Thus $u_2' = -1$ and $u_1' = x^{-1}$. Hence $u_2(x) = -x$ and $u_1(x) = \log(x)$.

The particular solution is therefore:

$$y(x) = x^{-1}\log(x) - x \cdot x^{-2} = x^{-1}\log(x) - x^{-1}$$

and the general solution is:

$$y(x) = x^{-1}\log(x) + Ax^{-1} + Bx^{-2}$$

(noting that as x^{-1} is a solution of the homogeneous, we can omit that part from the particular solution).

Problem 4 Let

$$A = \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ -2 & -6 & 5 & 1 & -8 \\ 2 & 6 & 2 & 6 & -2 \\ -1 & -3 & 0 & -2 & 2 \end{bmatrix}$$

Find bases for $\mathrm{Null}(A)$, $\mathrm{Col}(A)$, and $\mathrm{Row}(A)$. What is $\dim(\mathrm{Null}(A^T))$?

Solution:

We begin by row reducing the matrix:

$$\begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ -2 & -6 & 5 & 1 & -8 \\ 2 & 6 & 2 & 6 & -2 \\ -1 & -3 & 0 & -2 & 2 \end{bmatrix} \xrightarrow[\substack{\rho_3-2\rho_1 \\ \rho_4+\rho_1}]{\rho_2+2\rho_1} \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 6 & 6 & -8 \\ 0 & 0 & -2 & -2 & 5 \end{bmatrix}$$

$$\xrightarrow[\rho_4+2\rho_2]{\rho_3+2\rho_2} \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{\rho_4-4\rho_3} \begin{bmatrix} 1 & 3 & -2 & 0 & 3 \\ 0 & 0 & 1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

There are pivots in columns 1, 3, and 5 so a basis for the column space is:

$$\left\{ \begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 5 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 3 \\ -8 \\ -2 \\ 2 \end{bmatrix} \right\}$$

The free variables in columns 2 and 4 give us a basis for the null space as:

$$\left\{ \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} \right\}$$

The row space has basis the non-zero rows in the row reduced form, thus:

$$\left\{ \begin{bmatrix} 1 \\ 3 \\ -2 \\ 0 \\ 3 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 4 \end{bmatrix} \right\}$$

Lastly, the row space of A is the column space of A^T, then we have:

$$\dim \operatorname{Null}(A^T) + \dim \operatorname{Col}(A^T) = 4$$

whence $\dim \mathrm{Null}(A^T) = 1$.

Problem 5 For which values of a is the following family of vectors linearly independent? For each a where the family is linearly dependent, give a non-trivial linear dependency between the vectors.

$$\left\{ \begin{bmatrix} a \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix}, \begin{bmatrix} a \\ -1 \\ 0 \end{bmatrix} \right\}$$

Solution:

There are a variety of ways to solve this. Since the question asks for relations in the cases where the vectors are not linearly independent, let us start from an arbitrary relation:

$$c_1 \begin{bmatrix} a \\ -2 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix} + c_3 \begin{bmatrix} a \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The third row tells us that $c_1 + 3c_2 = 0$ and thus we can simplify this to:

$$c_2 \left(-3 \begin{bmatrix} a \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix} \right) + c_3 \begin{bmatrix} a \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

Equivalently,

$$c_2 \begin{bmatrix} -3a + 2 \\ 6 + a \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} a \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

The second row now tells us that $c_3 = (6 + a)c_2$ and thus the top row becomes:

$$c_2((-3a + 2) + (6 + a)a) = 0$$

This simplifies to:

$$c_2(a^2 + 3a + 2) = 0$$

So if $a = -1$ or $a = -2$ then there is a dependency, and if not then the vectors are linearly independent.

If $a = -1$, we pick $c_2 = 1$ and then $c_3 = 5$ and $c_1 = -3$. That is,

$$a = -1, \quad -3\begin{bmatrix} a \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix} + 5\begin{bmatrix} a \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

If $a = -2$ then putting $c_2 = 1$ we get $c_1 = -3$ as before and $c_3 = 4$. That is,

$$a = -2, \quad -3\begin{bmatrix} a \\ -2 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ a \\ 3 \end{bmatrix} + 4\begin{bmatrix} a \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Problem 6 Let

$$A = \begin{bmatrix} 3 & 7 & 5 \\ 4 & 1 & -10 \\ 0 & 1 & 2 \end{bmatrix}$$

Find an orthogonal basis for $\mathbb{R}^3$ that begins with an orthogonal basis for $\mathrm{Col}(A)$.

Solution:

There are a couple of ways to do this. One is to append the standard basis to the columns of the matrix and then apply Gram–Schmidt. Another is to apply Gram–Schmidt to the columns of the matrix and then compute an orthogonal basis for $\mathrm{Col}(A)^\perp = \mathrm{Null}(A^T)$.

Either way, we begin by applying Gram–Schmidt to the columns of the above matrix. We start with:

$$\vec{u}_1 = \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}$$

and set

$$\vec{u}_2 = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 7 \\ 1 \\ 1 \end{bmatrix} - \frac{25}{25}\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}$$

Then when we work with the third vector, we find:

$$\begin{aligned}
&\begin{bmatrix} 5 \\ -10 \\ 2 \end{bmatrix} - \frac{\begin{bmatrix} 5 \\ -10 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}}{\begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \bullet \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}} \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} - \frac{\begin{bmatrix} 5 \\ -10 \\ 2 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}}{\begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \bullet \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 5 \\ -10 \\ 2 \end{bmatrix} - \frac{52}{26} \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} - \frac{-25}{25} \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}
\end{aligned}$$

So an orthogonal basis for $\mathrm{Col}(A)$ is:

$$\left\{ \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix} \right\}$$

We only need one more vector for a basis for $\mathbb{R}^3$, so the simplest method is to solve:

$$\begin{bmatrix} 3 & 4 & 0 \\ 4 & -3 & 1 \end{bmatrix} \vec{x} = \vec{0}$$

Row reduction leads to:

$$\begin{bmatrix} 3 & 4 & 0 \\ 4 & -3 & 1 \end{bmatrix} \xrightarrow{3\rho_2} \begin{bmatrix} 3 & 4 & 0 \\ 12 & -9 & 3 \end{bmatrix} \xrightarrow{\rho_2 - 4\rho_1} \begin{bmatrix} 3 & 4 & 0 \\ 0 & -25 & 3 \end{bmatrix}$$

which has solution

$$\begin{bmatrix} -4 \\ 3 \\ 25 \end{bmatrix}$$

Thus the sought-after basis is:

$$\left\{ \begin{bmatrix} 3 \\ 4 \\ 0 \end{bmatrix}, \begin{bmatrix} 4 \\ -3 \\ 1 \end{bmatrix}, \begin{bmatrix} -4 \\ 3 \\ 25 \end{bmatrix} \right\}$$

Problem 7 The newly opened student cafeteria at the Antarctic University of Tropical Medicine offers a choice of three meals: a meat dish, a vegetarian dish, and a sandwich meal. Extensive research reveals that students select their meal based solely on what they ate the day before. There is a probability of 0.1 that a student will eat the same as the previous day. If a student did not have the meat dish yesterday, then the probability that they will choose the meat dish today is 0.3. If a student did have the meat dish yesterday, then they will choose the sandwich today with probability 0.3.

Set up the stochastic matrix for this situation and use it to find the proportions that the caterers should buy the meals in the long term.

Solution:

To set up the stochastic matrix, we need to assign an order to the preferences. We will list them as "meat", "vegetarian", "sandwich". The information from the description allows us to fill in the following data:

$$\begin{bmatrix} 0.1 & 0.3 & 0.3 \\ & 0.1 & \\ 0.3 & & 0.1 \end{bmatrix}$$

As it is a stochastic matrix, the columns must sum to 1 and this allows us to fill in the rest:

$$\begin{bmatrix} 0.1 & 0.3 & 0.3 \\ 0.6 & 0.1 & 0.6 \\ 0.3 & 0.6 & 0.1 \end{bmatrix}$$

We are asked about the "long term" solution, which means that we need to find the steady state solution. Letting P be the matrix above, we look for $\vec{q}$ such that $P\vec{q} = \vec{q}$. That is, $\vec{q}$ is a probability vector in $\operatorname{Null}(P - I)$. To find this, we row reduce $P - I$:

$$\begin{bmatrix} -0.9 & 0.3 & 0.3 \\ 0.6 & -0.9 & 0.6 \\ 0.3 & 0.6 & -0.9 \end{bmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_3} \begin{bmatrix} 0.3 & 0.6 & -0.9 \\ 0.6 & -0.9 & 0.6 \\ -0.9 & 0.3 & 0.3 \end{bmatrix}$$

$$\xrightarrow[\rho_3 + 3\rho_1]{\rho_2 - 2\rho_1} \begin{bmatrix} 0.3 & 0.6 & -0.9 \\ 0 & -2.1 & 2.4 \\ 0 & 2.1 & -2.4 \end{bmatrix}$$

$$\xrightarrow{\rho_3 + \rho_2} \begin{bmatrix} 0.3 & 0.6 & -.9 \\ 0 & -2.1 & 2.4 \\ 0 & 0 & 0 \end{bmatrix}$$

This has solution:

$$\begin{bmatrix} \frac{5}{7} \\ \frac{8}{7} \\ 1 \end{bmatrix}$$

Renormalising, we obtain

$$\frac{7}{20}\begin{bmatrix} \frac{5}{7} \\ \frac{8}{7} \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{4} \\ \frac{2}{5} \\ \frac{7}{20} \end{bmatrix}$$

Thus the caterers should buy the meals in the following proportions:

- $\frac{1}{4} = .25$ meat,
- $\frac{2}{5} = .4$ vegetarian,
- $\frac{7}{20} = .35$ sandwich.

Problem 8 Find the eigenvalues and eigenvectors (which might be complex) of the matrix

$$\begin{bmatrix} 0 & 0 & -1 \\ 1 & -2 & 2 \\ 1 & 0 & 0 \end{bmatrix}$$

Find the solution of the following system of differential equations with initial conditions $x_1 = 0, x_2 = 0, x_3 = 1$:

$$\begin{aligned} x_1' &= -x_3 \\ x_2' &= x_1 - 2x_2 + 2x_3 \\ x_3' &= x_1 \end{aligned}$$

Write your answer in terms of real functions.

Solution:

There is an obvious eigenvalue and eigenvector: -2 with $\begin{bmatrix}0\\1\\0\end{bmatrix}$. To find the others, we compute the characteristic polynomial:

$$\begin{vmatrix}-\lambda & 0 & -1\\ 1 & -2-\lambda & 2\\ 1 & 0 & -\lambda\end{vmatrix} = (-2-\lambda)\begin{vmatrix}-\lambda & -1\\ 1 & -\lambda\end{vmatrix} = (-2-\lambda)(\lambda^2+1)$$

Thus the other two eigenvalues are $\pm i$.

To find the (complex) eigenvectors, we row reduce as follows:

$$\begin{bmatrix}-i & 0 & -1\\ 1 & -2-i & 2\\ 1 & 0 & -i\end{bmatrix} \xrightarrow[\rho_3 - i\rho_1]{\rho_2 - i\rho_1} \begin{bmatrix}-i & 0 & -1\\ 0 & -2-i & 2+i\\ 0 & 0 & 0\end{bmatrix} \xrightarrow[\rho_2/(-2-i)]{i\rho_1} \begin{bmatrix}1 & 0 & -i0 & 1 & -10 & 0 & 0\end{bmatrix}$$

which has solution:

$$\begin{bmatrix}i\\1\\1\end{bmatrix}$$

The other eigenvector (for eigenvalue $-i$) will therefore be:

$$\begin{bmatrix}-i\\1\\1\end{bmatrix}$$

The system of differential equations is of the form $\vec{x}' = A\vec{x}$. Its general solution is therefore of the form:

$$\vec{x}(t) = c_1 e^{it}\begin{bmatrix}i\\1\\1\end{bmatrix} + c_2 e^{-it}\begin{bmatrix}-i\\1\\1\end{bmatrix} + c_3 e^{-2t}\begin{bmatrix}0\\1\\0\end{bmatrix}$$

To find c_1, c_2, and c_3 we look at the initial conditions. There, we have:

$$c_1\begin{bmatrix}i\\1\\1\end{bmatrix} + c_2\begin{bmatrix}-i\\1\\1\end{bmatrix} + c_3\begin{bmatrix}0\\1\\0\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix}$$

Thus $c_1 + c_2 = 1$ and $c_1 - c_2 = 0$, whence $c_1 = c_2 = 1/2$. Lastly, $c_3 = -(c_1 + c_2) = -1$. Hence the solution is:

$$\vec{x}(t) = \frac{1}{2}e^{it}\begin{bmatrix} i \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}e^{-it}\begin{bmatrix} -i \\ 1 \\ 1 \end{bmatrix} - e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

To write this in terms of real functions, we can gather terms as follows.

$$\vec{x}(t) = \frac{1}{2}\begin{bmatrix} ie^{it} - ie^{-it} \\ e^{it} + e^{-it} \\ e^{it} + e^{-it} \end{bmatrix} - e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -2\sin(t) \\ 2\cos(t) \\ 2\cos(t) \end{bmatrix} - e^{-2t}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Hence:

$$\vec{x}(t) = \begin{bmatrix} -\sin(t) \\ \cos(t) - e^{-2t} \\ \cos(t) \end{bmatrix}$$

Problem 9 A scientist records the following data for her experiment:

Control (x)	-2	-1	0	1	2
Reading (y)	5	4	2	4	15

The model for this data is a quadratic $y = ax^2 + bx + c$. Find the least-squares solution for a, b, c that best fits the data.

Solution:

The model is $y = ax^2 + bx + c$ so we obtain a system of linear equations when we try to fit this data. As a matrix equation, this is:

$$\begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 2 \\ 4 \\ 15 \end{bmatrix}$$

This does not have a solution, so we look for a least-squares solution by solving $A^T A\vec{a} = A^T\vec{y}$. First, we compute:

$$A^T A = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix}$$

and

$$A^T\vec{y} = \begin{bmatrix} 4 & 1 & 0 & 1 & 4 \\ -2 & -1 & 0 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 2 \\ 4 \\ 15 \end{bmatrix} = \begin{bmatrix} 88 \\ 20 \\ 30 \end{bmatrix}$$

We row reduce the augmented matrix:

$$\begin{bmatrix} 34 & 0 & 10 & 88 \\ 0 & 10 & 0 & 20 \\ 10 & 0 & 5 & 30 \end{bmatrix} \xrightarrow{\rho_1 \leftrightarrow \rho_3} \begin{bmatrix} 10 & 0 & 5 & 30 \\ 0 & 10 & 0 & 20 \\ 34 & 0 & 10 & 88 \end{bmatrix} \xrightarrow[\rho_2/10]{\rho_1/5} \begin{bmatrix} 2 & 0 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 34 & 0 & 10 & 88 \end{bmatrix} \xrightarrow{\rho_3 - 17\rho_1} \begin{bmatrix} 2 & 0 & 1 & 6 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & -7 & -14 \end{bmatrix}$$

which produces the solution:

$$\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$

Hence the best fit quadratic is $2x^2 + 2x + 2$.

Problem 10 Let A be an $m \times n$–matrix. Explain why $A^T A$ is a symmetric matrix. What size is it?

Use the spectral theorem to show that there is an orthogonal basis $\{\vec{v}_j\}$ for $\mathbb{R}^n$ such that $\{A\vec{v}_j\}$ is an orthogonal family (i.e. the vectors are pairwise orthogonal).

Solution:

To show that $A^T A$ is symmetric, we take its transpose: $(A^T A)^T$. The transpose of a product is the product of the transposes, with the order reversed: $(BC)^T = C^T B^T$. Thus $(A^T A)^T = A^T (A^T)^T$. Taking the transpose twice gets us back to where we started: $(B^T)^T = B$. Thus $(A^TA)^T = A^TA$. Hence A^TA is symmetric.

Since A is $m \times n$, A^T is $n \times m$. Hence A^TA is $n \times n$.

The spectral theorem says that there is an orthogonal basis for $\mathbb{R}^n$ consisting of eigenvectors of A^TA. Let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be such a basis. Let $\vec{u}_j = A\vec{v}_j$. Then for $i \neq j$,

$$\vec{u}_i \bullet \vec{u}_j = (A\vec{v}_i) \bullet (A\vec{v}_j) = (A\vec{v}_i)^T(A\vec{v}_j) = \vec{v}_i^T A^T A \vec{v}_j$$

Now $\vec{v}_j$ is an eigenvector of A^TA, say with eigenvalue λ_j. So:

$$\vec{v}_i^T A^T A \vec{v}_j = \vec{v}_i^T (\lambda_j \vec{v}_j) = \lambda_j \vec{v}_i^T \vec{v}_j = \lambda_j \vec{v}_i \bullet \vec{v}_j.$$

As $i \neq j$, $\vec{v}_i \bullet \vec{v}_j = 0$. Hence $\vec{u}_i \bullet \vec{u}_j = 0$ and thus $\{\vec{u}_1, \dots, \vec{u}_n\}$ is an orthogonal family.