NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4110/TMA4115 Matematikk 3**

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Examination time (from–to):

Permitted examination support material: C: Simple Calculator (Casio fx-82ES PLUS, Citizen SR-270X, Citizen SR-270X College, or Hewlett Packard HP30S), Rottmann: Matematisk formelsamling.

Other information:
Give reasons for all answers, ensuring that it is clear how the answers have been reached. Each of the 8 problems has the same weight.

Language: English

Number of pages: 3

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

a) Compute $\left(\frac{1}{-1+i\sqrt{3}}\right)^6$.

b) Use the polar form $z = r \cdot e^{i\theta}$ to find all complex numbers z satisfying

$$2z^2 - \bar{z}^3 = 0.$$

Draw the solutions in the complex plane.

Problem 2

Find the unique function $y(t)$ that solves the initial value problem

$$\frac{1}{4}y'' - y' + y = 5e^{2t} + 1, \; y(0) = 1, \; y'(0) = 1.$$

Problem 3

Consider the following system of differential equations

$$\mathbf{x}' = A\mathbf{x} \text{ with } A = \begin{pmatrix} 1 & -1 \\ 2 & 4 \end{pmatrix}. \qquad (1)$$

a) Diagonalize the matrix A: find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

b) We set the change of variable $\mathbf{y} = P^{-1}\mathbf{x}$. Which differential equation is satisfied by $\mathbf{y}$?

c) Find the unique solution of the system (1) which satisfies $\mathbf{x}(0) = \begin{pmatrix} 1 \\ 2 \end{pmatrix}$.

Problem 4 Let $T\colon \mathbb{R}^4 \to \mathbb{R}$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = x - y + 2z - 2w.$$

Find an orthogonal basis for the null space of T.

Problem 5

Let $A = \begin{bmatrix} a & a-1 & a \\ a-1 & 1 & 0 \\ a & 0 & a \end{bmatrix}$

a) Determine the rank of A for every real number a.

b) Determine all real numbers a and b such that the linear system

$$A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 1 \end{bmatrix}$$

has infinitely many solutions.

Problem 6

Chess player Magnus can either win, draw or lose a game. His coach observes the following pattern in Magnus' games:

- After a win, there is a 70% chance that he wins the next game as well and only a 10% chance that he loses the next game.
- After a draw, there is an 80% chance that the next game is a draw as well, but only a 10% chance that he wins the next game.
- After losing a game, there is a 30% chance that he wins the next game and a 30% chance for a draw in the next game.

After many games of this pattern, what is the most likely outcome of Magnus' next game? (Give the probabilities for the three possible outcomes.)

Problem 7

Find the equation $y = ax^2 + bx + c$ which best fits the data points $(-2,6)$, $(-1,6)$, $(0,-2)$, $(1,2)$ and $(2,3)$.

Problem 8

Suppose that A is an $n \times n$ matrix for which A^2 is the zero matrix, i.e. the $n \times n$ matrix with zeroes in every position.

a) Prove that A is not invertible.

b) Show that the only eigenvalue of A is 0.

c) Give a particular example of such an A that is not the zero matrix.
(Hint: consider the 2×2-case)