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Department of Mathematical Sciences

Examination paper for **TMA4110/TMA4115 Matematikk 3**

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Other information:
Give reasons for all answers, ensuring that it is clear how the answers have been reached. Each of the 8 problems has the same weight.

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Problem 1

a) Compute $\left(\frac{1}{-1+i\sqrt{3}}\right)^6$.

b) Use the polar form $z = r \cdot e^{i\theta}$ to find all complex numbers z satisfying

$$2z^2 - \bar{z}^3 = 0.$$

Draw the solutions in the complex plane.

Solution

a) We compute:

$$\left(\frac{1}{-1+i\sqrt{3}}\right)^6 = \left(\frac{-1-i\sqrt{3}}{4}\right)^6 = \left(\frac{1}{2}\cdot\frac{1+i\sqrt{3}}{2}\right)^6 = \left(\frac{1}{2}e^{i\pi/3}\right)^6 = \frac{1}{2^6}e^{6i\pi/3} = \frac{1}{64}.$$

b) Setting $z = r \cdot e^{i\theta}$, we get

$$2z^2 = r^2 \cdot e^{i2\theta} \text{ and } \bar{z}^3 = r^3 \cdot e^{-i3\theta}.$$

Hence z must satisfy $2r^2 = r^3$, i.e. $r = 0$ or $r = 2$, and $2\theta = -3\theta + 2\pi k$ for an integer k. For $r = 0$, we have $z = 0$ as a solution. For $r = 2$, it suffices to consider $k = 0, 1, 2, 3, 4$. The solutions are

$$z = 0,\ z = 2,\ z = 2e^{i2\pi/5},\ z = 2e^{i4\pi/5},\ z = 2e^{i6\pi/5}, \text{ and } z = 2e^{i8\pi/5}.$$

Problem 2

Solve the initial value problem

$$\frac{1}{4}y'' - y' + y = 5e^{2t} + 1,\ y(0) = 1,\ y'(0) = 1.$$

Solution

The characteristic equation is $\lambda^2 - 4\lambda + 4\lambda = 0$ and has a double real root $\lambda = 2$. Hence c_1te^{2t} and c_2e^{2t} are solutions of the associated homogeneous equation. Since the first summand of the forcing term, $5e^{2t}$, is a solution of the homogeneous equation, we need a trial solution of the form $at^2e^{2t} + b$ to find a particular solution for the inhomogeneous equation. After solving for a and b, we get that the general solution is

$$e^{2t}(10t^2 + c_1t + c_2) + 1.$$

Solving for the initial conditions we get $y(t) = e^{2t}(10t^2 + t) + 1$.

Problem 3

Consider the following system of differential equations

$$\mathbf{x}' = A\mathbf{x} \text{ with } A = \begin{bmatrix} 1 & -1 \\ 2 & 4 \end{bmatrix}. \tag{1}$$

a) Diagonalize the matrix A: find an invertible matrix P such that $P^{-1}AP$ is a diagonal matrix.

b) We set the change of variable $\mathbf{y} = P^{-1}\mathbf{x}$. Which differential equation is satisfied by $\mathbf{y}$?

c) Find the unique solution of the system (1) which satisfies $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Solution

a) We look at the characteristic polynomial of A: $\lambda^2 - 5\lambda + 6 = (\lambda - 2)(\lambda - 3)$. Hence the eigenvalues of A are 2 and 3. A choice of corresponding eigenvectors is given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \end{bmatrix}$. A matrix P we are looking for has these eigenvectors as columns. Hence one possible choice is

$$P = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \text{ with } P^{-1} = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix}, \text{ and set } D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Then we have $D = P^{-1}AP$.

b) The new $\mathbf{y}$ satisfies the system of differential equations

$$\begin{cases} y_1' = 2y_1 \\ y_2' = 3y_2. \end{cases} \tag{2}$$

c) The system (2) is solved by $\mathbf{y}(t) = \begin{bmatrix} c_1 e^{2t} \\ c_2 e^{3t} \end{bmatrix}$. The initial condition for $\mathbf{y}$ is

$$\mathbf{y}(0) = P^{-1}\mathbf{x}(0) = \begin{bmatrix} 2 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

This implies $c_1 = 4$ and $c_2 = -3$. Hence the unique solution we are looking for is

$$\mathbf{x}(t) = P\mathbf{y}(t) = \begin{bmatrix} 1 & 1 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} 4e^{2t} \\ -3e^{3t} \end{bmatrix} = \begin{bmatrix} 4e^{2t} - 3e^{3t} \\ -4e^{2t} + 6e^{3t} \end{bmatrix}.$$

Problem 4 Let $T\colon \mathbb{R}^4 \to \mathbb{R}$ be the linear transformation given by

$$T\left(\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}\right) = x - y + 2z - 2w.$$

Find an orthogonal basis for the null space of T.

Solution

The null space is given by all vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ in $\mathbb{R}^4$ such that $x - y + 2z - 2w = 0$.

This is an equation with four variables. Hence there are three free variables to choose. For example, we could y, z and w as free variables.

For $y = 1$ and $z = w = 0$, we get $x = 1$ and a vector $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}$.

For $z = 1$ and $y = w = 0$, we get $x = -2$ and a vector $\mathbf{u}_2 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

For $w = 1$ and $y = z = 0$, we get $x = 2$ and a vector $\mathbf{u}_3 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

The vectors $\mathbf{u}_1$, $\mathbf{u}_2$, $\mathbf{u}_3$ are obviously linearly independent (look at what happens for the last three coordinates). Since the dimension of the null space of T is three, $\{\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3\}$ is a basis of the null space.

It remains to make the basis orthogonal. We do this using the Gram-Schmidt process. We set $\mathbf{v}_1 := \mathbf{u}_1$ and define

$$\mathbf{v}_2 := \mathbf{u}_2 - \frac{\mathbf{u}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \cdot \mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{-2}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix},$$

and

$$\mathbf{v}_3 := \mathbf{u}_3 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \cdot \mathbf{v}_1 - \frac{\mathbf{u}_3 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2} \cdot \mathbf{v}_2 = \begin{bmatrix} 2 \\ 0 \\ 0 \\ 1 \end{bmatrix} - \frac{2}{2} \cdot \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{-2}{3} \cdot \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -1/3 \\ 2/3 \\ 1 \end{bmatrix}.$$

An orthogonal basis for the null space of T is given by

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1/3 \\ -1/3 \\ 2/3 \\ 1 \end{bmatrix}.$$

Problem 5

Let $A = \begin{bmatrix} a & a-1 & a \\ a-1 & 1 & 0 \\ a & 0 & a \end{bmatrix}$

a) Determine the rank of A for every real number a.

b) Determine all real numbers a and b such that the linear system

$$A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b \\ 0 \\ 1 \end{bmatrix}$$

has infinitely many solutions.

Solution

a) Use Gaussian elimination to determine the rank:

$$\begin{bmatrix} a & a-1 & a \\ a-1 & 1 & 0 \\ a & 0 & a \end{bmatrix} \rightsquigarrow \begin{bmatrix} a & 0 & a \\ a-1 & 1 & 0 \\ a & a-1 & a \end{bmatrix} \overset{a\neq 0}{\rightsquigarrow} \begin{bmatrix} a & 0 & a \\ 0 & 1 & 1-a \\ 0 & a-1 & 0 \end{bmatrix}$$
$$\rightsquigarrow \begin{bmatrix} a & 0 & a \\ 0 & 1 & 1-a \\ 0 & 0 & (1-a)^2 \end{bmatrix}$$

From the row echelon form we see that $\operatorname{rank} A = 3$ for all real numbers a with $a \neq 0$ and $a \neq 1$. If $a = 1$ the row echelon form shows that $\operatorname{rank} A = 2$. Now for $a = 0$ the original matrix is $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ whence obviously a rank 2 matrix. (Note that we can not argue with the above echelon form for $a = 0$ since we had to divide by a in computing it).

b) From part (a) we know already that $\operatorname{rank} A = 3$ if $a \neq 0$ and $a \neq 1$. In these cases A has full rank, is invertible and the equation $A\mathbf{x} = \mathbf{b}$ always has a unique solution. Hence we only have to check $a = 1$ and $a = 0$.

First case, $a = 0$: then $\begin{bmatrix} 0 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ whence the linear system is inconsistent (as the last row of the matrix is a zero row and $\begin{bmatrix} b \\ 0 \\ 1 \end{bmatrix}$ is non-zero in the last row.

Second case, $a = 1$: then $A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Using Gaussian elimination on the linear system we see

$$\begin{bmatrix} 1 & 0 & 1 & b \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 1 & 0 & 1 & b \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1-b \end{bmatrix}$$

Thus the system is only consistent if $b = 1$ and in this case there are infinitely many solutions. Summing up, the linear system has infinitely many solutions if and only if for $a = 1$ and $b = 1$.

Problem 6

Chess player Magnus can either win, draw or lose a game. His coach observes the following pattern in Magnus' games:

- After a win, there is a 70% chance that he wins the next game as well and only a 10% chance that he loses the next game.
- After a draw, there is an 80% chance that the next game is a draw as well, but only a 10% chance that he wins the next game.
- After losing a game, there is a 30% chance that he wins the next game and a 30% chance for a draw in the next game.

After many games of this pattern, what is the most likely outcome of Magnus' next game? (Give the probabilities for the three possible outcomes.)

Solution

The stochastic matrix which describes the probability of a win, draw or loss in the next game is

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.3 \\ 0.2 & 0.8 & 0.3 \\ 0.1 & 0.1 & 0.4 \end{bmatrix}.$$

We want to find the stationary vector, it is a probability vector (i.e. a vector whose entries are not negative and add up to 1) which satisfies $Av = v$. Thus we need to solve the system of linear equations with matrix $A - I$. After multiplying by 10, Gauss elimination gives

$$10(A-I) = \begin{bmatrix} -3 & 1 & 3 \\ 2 & -2 & 3 \\ 1 & 1 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 4 & -15 \\ 0 & -4 & 15 \\ 1 & 1 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & -4 & 15 \\ 1 & 1 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & -15/4 \\ 1 & 0 & -9/4 \end{bmatrix}$$

The solutions satisfy $x_2 = 15/4x_3$ and $x_1 = 9/4x_3$. Choosing $x_3 = 4$ yields $x_1 = 9$ and $x_2 = 15$. But since we are looking for a probability vector, the sum of the coordinates must be 1. Hence we divide each coordinate by $9+15+4 = 28$ and get the stationary probability vector

$$v = \begin{bmatrix} 9/28 \\ 15/28 \\ 4/28 \end{bmatrix}.$$

Thus the most likely outcome of the game is a draw with a probability of $15/28 \approx 54\%$. The probability for a win is $9/28 \approx 32\%$ and the probability for a loss is $4/28 \approx 14\%$.

Problem 7

Find the equation $y = ax^2 + bx + c$ which best fits the data points $(-2,6)$, $(-1,6)$, $(0,-2)$, $(1,2)$ and $(2,3)$.

Solution

We are looking for the least square solution of the system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 4 & -2 & 1 \\ 1 & -1 & 1 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \\ 4 & 2 & 1 \end{bmatrix}, \; \mathbf{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}, \; \mathbf{b} = \begin{bmatrix} 6 \\ 6 \\ -2 \\ 2 \\ 3 \end{bmatrix}.$$

To find the solution we solve the system $A^T A\mathbf{x} = A^T\mathbf{b}$ which is

$$\begin{bmatrix} 34 & 0 & 10 \\ 0 & 10 & 0 \\ 10 & 0 & 5 \end{bmatrix} \begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 44 \\ -10 \\ 15 \end{bmatrix}.$$

Gauss elimination gives

$$\begin{bmatrix} 34 & 0 & 10 & 44 \\ 0 & 10 & 0 & -10 \\ 10 & 0 & 5 & 15 \end{bmatrix} \to \begin{bmatrix} 10 & 0 & 5 & 15 \\ 0 & 10 & 0 & -10 \\ 34 & 0 & 10 & 44 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 34 & 0 & 10 & 44 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & 1 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & -7 & -7 \end{bmatrix}$$

The solution is $a = 1$, $b = -1$ and $c = 1$. Thus the best fitting equation is

$$y = x^2 - x + 1.$$

Problem 8

Suppose that A is an $n \times n$ matrix for which A^2 is the zero matrix, i.e. the $n \times n$ matrix with zeroes in every position.

a) Prove that A is not invertible.

b) Show that the only eigenvalue of A is 0.

c) Give a particular example of such an A that is not the zero matrix.
(Hint: consider the 2×2-case)

Solution

a) Suppose A has an inverse. Then (right) multiply the equation $A^2 = 0$ by A^{-1} to obtain $AAA^{-1} = 0$, and hence $A = 0$, a contradiction as the zero matrix is not invertible.

b) From the first part, A is not invertible and hence (by the invertible matrix theorem) has zero as an eigenvalue. Now suppose that λ is a general eigenvalue with eigenvector $\mathbf{v}$. We have

$$A(A\mathbf{v}) = A(\lambda)\mathbf{v} = \lambda A\mathbf{v} = \lambda^2\mathbf{v}.$$

However, $A^2\mathbf{v} = 0$ as $A^2 = 0$, and hence $\lambda^2\mathbf{v} = 0$. As the zero vector can never be an eigenvector, this implies that $\lambda^2 = 0$ and hence that $\lambda = 0$. So zero is the only eigenvalue.

c) The simplest examples are

$$\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$$