NTNU – Trondheim
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Department of Mathematical Sciences

Examination paper for **TMA4115 Matematikk 3**

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Examination time (from–to): 09:00–13:00

Permitted examination support material: C: Simple Calculator (Casio fx-82ES PLUS, Citizen SR-270X, Citizen SR-270X College, or Hewlett Packard HP30S), Rottmann: Matematiske formelsamling.

Other information:
Give reasons for all answers, ensuring that it is clear how the answers have been reached. Each of the 10 problems has the same weight.

Language: English

Number of pages: 4

Number pages enclosed: 0

Checked by:

Date Signature

Problem 1

a) For $z = (-1 + i\sqrt{3})$, compute z^3 and $|z|^6$.

b) Find all complex numbers z with $z^3 = 8i$ and draw them in the complex plane.

Problem 2

Consider the inhomogeneous differential equation

$$y'' + 6y' + 9y = \cos t \tag{1}$$

a) Find the general solution of the associated homogeneous equation.

b) Find a particular solution of (1).

c) Find the unique solution of (1) that satisfies $y(0) = y'(0) = 0$.

Problem 3

Let a be a real number and A be the matrix $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

a) Find a fundamental set of real solutions to the differential equation $\mathbf{x}' = \mathbf{A}\mathbf{x}$.

b) Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Problem 4

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$ be vectors in $\mathbb{R}^3$.

a) Write the vector $\mathbf{p} = \begin{bmatrix} 2 \\ 4 \\ -10 \end{bmatrix}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

b) Can you write the vector $\mathbf{q} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$?

c) Are $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ linearly independent?

d) What is the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 6 & -1 \end{bmatrix}$?

Problem 5

a) Find the inverse of the matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix}$.

b) Let $T \colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2x_2 \\ x_3 \\ 4x_1 + 2x_2 \end{bmatrix}.$$

Is T one-to-one?

Problem 6

Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 5 & 4 \\ 3 & 6 & -1 & 8 & 5 \\ 5 & 4 & 8 & -1 & 1 \end{bmatrix}.$$

a) Bring A into row echelon form.

b) Find a basis for $\mathrm{Col}(A)$ and determine the rank of A.

c) Determine the dimension of $\mathrm{Nul}(A)$.

d) Determine the dimensions of $\mathrm{Row}(A)$ and of $\mathrm{Nul}(A^T)$.

Problem 7

The temperature in Bymarka during winter season can be either above, equal to or below 0° Celsius. Trondheim's ski club observes the following fluctuation of temperatures from one day to the next:

- If the temperature has been above 0°, there is a 70% chance that it will be above and a 10% chance that it will be below 0° the next day.
- If the temperature has been equal to 0°, there is a 10% chance that it will be above and a 10% chance that it will be below 0° the next day.
- If the temperature has been below 0°, there is a 10% chance that it will be above and a 70% chance that it will be below 0° the next day.

After many days of this pattern in the winter, for what temperature should a skier prepare his/her skies? (Give the probabilities for the three possible temperatures.)

Problem 8

Find the equation $y = mx + c$ of the line that best fits the data points $(0,4)$, $(1,-1)$, $(2,1)$, $(3,-3)$ and $(4,-1)$.

Problem 9

Let A be the matrix $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $\mathbf{u}$ be the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

a) Verify that 2 is an eigenvalue of A and that $\mathbf{u}$ is an eigenvector of A (possibly with an eigenvalue different from 2).

b) Find all the eigenvalues of A and a basis for each eigenspace of A.

c) Is A orthogonally diagonalizable? If so, orthogonally diagonalize A.

Problem 10

Let $W \subseteq \mathbb{R}^n$ be a subspace and $W^\perp$ be its orthogonal complement.

a) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$.

b) Let $\mathbf{w}$ be a vector which lies both in W and in $W^\perp$ (i.e. $\mathbf{w} \in W \cap W^\perp$). Show that this implies $\mathbf{w} = \mathbf{0}$.

c) Let $\{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ be a basis of W and let $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ be a basis of $W^\perp$. Show that $\{\mathbf{w}_1, \ldots, \mathbf{w}_r, \mathbf{v}_1, \ldots, \mathbf{v}_s\}$ is a basis of $\mathbb{R}^n$.