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Examination paper for **TMA4115 Matematikk 3**

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Give reasons for all answers, ensuring that it is clear how the answers have been reached. Each of the 10 problems has the same weight.

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Problem 1

a) For $z = (-1 + i\sqrt{3})$, compute z^3 and $|z|^6$.

b) Find all complex numbers z with $z^3 = 8i$ and draw them in the complex plane.

Solution

a) The polar form of $(-1 + i\sqrt{3})$ is $2e^{i2\pi/3}$. Hence

$$(-1+i\sqrt{3})^3 = (2e^{i2\pi/3})^3 = 2^3 e^{(i2\pi/3)3)} = 8e^{i2\pi} = 8 \text{ and } |z|^6 = |z^3| \cdot |z^3| = 64.$$

b) The polar form of $8i$ is $8e^{i\pi/2} = 2^3 e^{i\pi/2}$. For $z = re^{i\theta}$, $z^3 = 8i$ becomes

$$2^3 e^{i\pi/2} = (re^{i\theta})^3 = r^3 e^{i3\theta}.$$

This holds if and only if $r = 2$ and $\theta = \pi/6 + (2\pi/3)k$ for an integer k. It suffices to loook at the integers $k = 0$, $k = 1$ and $k = 2$. These yield the solutions

$$z_0 = 2e^{i\pi/6} = \sqrt{3} + i, \; z_1 = 2e^{i5\pi/6} = -\sqrt{3} + i \text{ and } z_2 = 2e^{i3\pi/2} = -2i.$$

Problem 2

Consider the inhomogeneous differential equation

$$y'' + 6y' + 9y = \cos t \tag{1}$$

a) Find the general solution of the associated homogeneous equation.

b) Find a particular solution of (1).

c) Find the unique solution of (1) that satisfies $y(0) = y'(0) = 0$.

Solution

a) The homogeneous equation is $y'' + 6y' + 9y = 0$. The characteristic equation is $\lambda^2 + 6\lambda + 9 = 0$, which has a double solution $\lambda = -3$. Therefore, a fundamental system of solutions is $y_1 = e^{-3t}$ and $y_2 = te^{-3t}$. The general solution is

$$y_h = c_1 e^{-3t} + c_2 te^{-3t}.$$

b) We can look for a particular solution using undetermined coefficients, trying

$$\begin{aligned} y_p &= a\cos t + b\sin t \\ y_p' &= -a\sin t + b\cos t \\ y_p'' &= -a\cos t - b\sin t. \end{aligned}$$

By substituting, we get that y_p is a solution if and only if

$$(-a + 6b + 9a)\cos t + (-b - 6a + 9b)\sin t = \cos t.$$

Therefore we have a system $8a + 6b = 1$ and $-6a + 8b = 0$. The solution is $a = 4/50$ and $b = 3/50$. So $y_p = (1/50)(4\cos t + 3\sin t)$.

c) From the two previous questions we know that the general solution is

$$y = c_1 e^{-3t} + c_2 te^{-3t} + (1/50)(4\cos t + 3\sin t).$$

Setting $y(0) = 0$ we get $c_1 = -4/50$. Deriving, we get

$$y' = -3c_1 e^{-3t} - 3c_2 te^{-3t} + c_2 e^{-3t} + 1/50(3\cos t - 4\sin t).$$

Hence $y'(0) = 0$ means $-3c_1 + c_2 + 3/50 = 0$, that is $c_2 = -15/50$. In the end, the solution we want is

$$y = (1/50)\Big((-4 - 15t)e^{-3t} + 4\cos t + 3\sin t\Big).$$

Problem 3 Let a be a real number and A be the matrix $\begin{bmatrix} 0 & a \\ -a & 0 \end{bmatrix}$.

a) Find a fundamental set of real solutions to the differential equation $\mathbf{x}' = \mathbf{Ax}$.

b) Solve the initial value problem $\mathbf{x}' = A\mathbf{x}$, where $\mathbf{x}(0) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$.

Solution

a) The characteristic equation is $\lambda^2 + a^2 = 0$, hence $\lambda = \pm ai$. We need an eigenvector corresponding to one of the eigenvalues. Take $\lambda = ai$. Then we must solve

$$a \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0$$

as we know the matrix is not invertible we can look at the first row only, and we get $v_2 = iv_1$. So a possible eigenvector is $\mathbf{v} = \begin{bmatrix} 1 \\ i \end{bmatrix}$.

A fundamental set of real solutions is then given by the functions $\mathbf{x}_1(t) = \mathrm{Re}(\mathbf{y}(t))$ and $\mathbf{x}_2(t) = \mathrm{Im}(\mathbf{y}(t))$, where $\mathbf{y}(t) = \mathbf{v}e^{\lambda t}$. We compute

$$\begin{aligned} \mathbf{y} &= \begin{bmatrix} 1 \\ i \end{bmatrix} e^{ait} \\ &= \begin{bmatrix} 1 \\ i \end{bmatrix} (\cos at + i \sin at) \\ &= \begin{bmatrix} \cos at + i \sin at \\ -\sin at + i \cos at \end{bmatrix} \end{aligned}$$

We therefore have $\mathbf{x}_1(t) = \begin{bmatrix} \cos at \\ -\sin at \end{bmatrix}$ and $\mathbf{x}_2(t) = \begin{bmatrix} \sin at \\ \cos at \end{bmatrix}$.

b) The solution to the initial value problem is then $\mathbf{x}(t) = c_1\mathbf{x}_1 + c_2\mathbf{x}_2$. We find the constants from $\mathbf{x}(0) = c_1\mathbf{x}_1(0) + c_2\mathbf{x}_2(0)$. This linear system is immediately solvable as $\mathbf{x}_1(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{x}_2(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. We have $c_1 = 2$ and $c_2 = 1$, and the solution to the initial value problem is

$$\mathbf{x}(t) = \begin{bmatrix} 2\cos at + \sin at \\ -2\sin at + \cos at \end{bmatrix}.$$

Problem 4

Let $\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ 6 \\ -1 \end{bmatrix}$ be vectors in $\mathbb{R}^3$.

a) Write the vector $\mathbf{p} = \begin{bmatrix} 2 \\ 4 \\ -10 \end{bmatrix}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

b) Can you write the vector $\mathbf{q} = \begin{bmatrix} 2 \\ 5 \\ 6 \end{bmatrix}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$?

c) Are $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ linearly independent?

d) What is the determinant of the matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 6 & -1 \end{bmatrix}$?

Solution

a) In order to find scalars x_1, x_2, x_3 with $x_1\mathbf{u} + x_2\mathbf{v} + x_3\mathbf{w} = \mathbf{p}$, we need to solve the linear system $A\mathbf{x} = \mathbf{p}$ with

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 1 & 6 & -1 \end{bmatrix}.$$

We do this by row operations on the augmented matrix

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 6 & 4 \\ 1 & 6 & -1 & -10 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & -4 & -12 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 5 & 0 & -7 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & -1 & -3 \end{bmatrix}.$$

Hence we can choose x_3 as a free variable, for example $x_3 = 1$. Then we get $x_2 = -2$ and $x_1 = 3$. Hence $\mathbf{p} = 3\mathbf{u} - 2\mathbf{v} + \mathbf{w}$.

b) When we try the same for $\mathbf{q}$, we get

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 2 & 4 & 6 & 5 \\ 1 & 6 & -1 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 0 & 0 & 1 \\ 0 & 4 & -4 & 4 \end{bmatrix}.$$

The second row corresponds to the false assertion $0 = 1$. Hence there is no solution to $A\mathbf{x} = \mathbf{q}$, and we cannot write $\mathbf{q}$ as a linear combination of $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$.

c) They are not linearly independent. There are several ways to see that. For example, we can deduce from the calculation in a) that $(-5)\mathbf{u} + \mathbf{v} + \mathbf{w} = \mathbf{0}$; or we use the second or third argument in d).

d) The determinant of A is 0. There are many ways to see that. For example, we have just observed that the columns of A (which are $\mathbf{u}$, $\mathbf{v}$ and $\mathbf{w}$) are linearly dependent; or we saw in a) that A is row equivalent to a matrix with a row with only zero entries; or we showed in b) that the linear transformation

$$A\colon \mathbb{R}^3 \to \mathbb{R}^3,\ \mathbf{x} \mapsto A\mathbf{x}$$

is not onto. Hence A is not invertible and $\det A = 0$.

Problem 5

a) Find the inverse of the matrix $A = \begin{bmatrix} 2 & 2 & 0 \\ 0 & 0 & 1 \\ 4 & 2 & 0 \end{bmatrix}$.

b) Let $T\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the linear transformation defined by

$$T\left(\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 + 2x_2 \\ x_3 \\ 4x_1 + 2x_2 \end{bmatrix}.$$

Is T one-to-one?

Solution

a) To find A^{-1} we perform row operations on A to reduce it to the identity matrix:

$$\begin{bmatrix} 2 & 2 & 0 & | & 1 & 0 & 0 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \\ 4 & 2 & 0 & | & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 & | & 1 & 0 & 0 \\ 4 & 2 & 0 & | & 0 & 0 & 1 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 & | & 1 & 0 & 0 \\ 0 & -2 & 0 & | & -2 & 0 & 1 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \end{bmatrix}$$

$$\to \begin{bmatrix} 2 & 0 & 0 & | & -1 & 0 & 1 \\ 0 & -2 & 0 & | & -2 & 0 & 1 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & | & -1/2 & 0 & 1/2 \\ 0 & 1 & 0 & | & 1 & 0 & -1/2 \\ 0 & 0 & 1 & | & 0 & 1 & 0 \end{bmatrix}.$$

Hence $A^{-1} = \begin{bmatrix} -1/2 & 0 & 1/2 \\ 1 & 0 & -1/2 \\ 0 & 1 & 0 \end{bmatrix}$.

b) T is one-to-one. There are at least two ways to see this. First, we could observe that A is the standard matrix of T (recall: the columns of the standard matrix are the images of the standard basis vectors e_1, e_2, e_3 under T). Since A is invertible, A is, in particular, also one-to-one. Hence T is one-to-one, since $\mathbf{0} = T(\mathbf{x}) = A\mathbf{x}$ implies $\mathbf{x} = \mathbf{0}$.

Second, we could just set $T(\mathbf{x}) = \mathbf{0}$. This can only be true if all three equations $2x_1 + 2x_2 = 0$, $x_3 = 0$ and $4x_1 + 2x_2 = 0$ hold. But this is true only if $x_1 = x_2 = x_3 = 0$. Hence $T(\mathbf{x}) = \mathbf{0}$ implies $\mathbf{x} = \mathbf{0}$ and T is one-to-one.

Problem 6

Let A be the matrix

$$A = \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 5 & 4 \\ 3 & 6 & -1 & 8 & 5 \\ 5 & 4 & 8 & -1 & 1 \end{bmatrix}.$$

a) Bring A into row echelon form.

b) Find a basis for $\mathrm{Col}(A)$ and determine the rank of A.

c) Determine the dimension of $\mathrm{Nul}(A)$.

d) Determine the dimensions of $\mathrm{Row}(A)$ and of $\mathrm{Nul}(A^T)$.

Solution

a) We use row operations to bring A into echelon form:

$$\begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 2 & 4 & -1 & 5 & 4 \\ 3 & 6 & -1 & 8 & 5 \\ 5 & 4 & 8 & -1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & -6 & 8 & -16 & -4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 & 3 & 1 \\ 0 & -6 & 8 & -16 & -4 \\ 0 & 0 & -1 & -1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

b) We observe that the first three columns of A are pivot columns. Hence the rank of A is 3, and the vectors $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 5 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ -1 \\ 8 \end{bmatrix}$ form a basis of $\mathrm{Col}(A)$.

c) The dimension of $\mathrm{Nul}(A)$ is 2. There are several ways to see this. One way is to observe that A has two non-pivot columns. Hence there are two free variables in the linear system corresponding to A. Another way is to use the Rank Theorem which tells us:

$$\text{number of columns of } A = \dim \mathrm{Col}(A) + \dim \mathrm{Nul}(A)$$

Hence $\dim \mathrm{Nul}(A) = 5 - 3 = 2$.

d) The row echelon form of A tells us that A has three linearly independent rows. The dimension of the row space is therefore 3.

The dimension of $\mathrm{Nul}(A^T)$ can be computed using the rank theorem and the fact $\mathrm{Col}(A^T) = \mathrm{Row}(A)$:

$$\text{number of rows of } A = \dim \mathrm{Col}(A^T) + \dim \mathrm{Nul}(A^T).$$

Hence $\dim \mathrm{Nul}(A^T) = 4 - 3 = 1$.

Problem 7

The temperature in Bymarka during winter season can be either above, equal to or below 0° Celsius. Trondheim's ski club observes the following fluctuation of temperatures from one day to the next:

- If the temperature has been above 0°, there is a 70% chance that it will be above and a 10% chance that it will be below 0° the next day.
- If the temperature has been equal to 0°, there is a 10% chance that it will be above and a 10% chance that it will be below 0° the next day.
- If the temperature has been below 0°, there is a 10% chance that it will be above and a 70% chance that it will be below 0° the next day.

After many days of this pattern in the winter, for what temperature should a skier prepare his/her skies? (Give the probabilities for the three possible temperatures.)

Solution

The stochastic matrix which describes the probability of the temperature being above, equal or below 0° is

$$A = \begin{bmatrix} 0.7 & 0.1 & 0.1 \\ 0.2 & 0.8 & 0.2 \\ 0.1 & 0.1 & 0.7 \end{bmatrix}.$$

We want to find the stationary vector, it is a probability vector (i.e. a vector whose entries are not negative and add up to 1) which satisfies $Av = v$. Thus we need to solve the system of linear equations with matrix $A - I$. After multiplying by 10, Gauss elimination gives

$$10(A-I) = \begin{bmatrix} -3 & 1 & 1 \\ 2 & -2 & 2 \\ 1 & 1 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -3 \\ 2 & -2 & 2 \\ -3 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -3 \\ 0 & -4 & 8 \\ 0 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -3 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}$$

The solutions satisfy $x_2 = 2x_3$ and $x_1 = -x_2 + 3x_3 = x_3$. Choosing $x_3 = 0.25$ yields $x_1 = 0.25$ and $x_2 = 0.5$. Hence the stationary probability vector is

$$v = \begin{bmatrix} 0.25 \\ 0.5 \\ 0.25 \end{bmatrix}.$$

Thus the most likely case is that the temperature is $0^\circ\ C$ with a 50% chance. The probability for a temperature above $0^\circ\ C$ is 25% and the probability for a temperature below $0^\circ\ C$ is also 25%.

Problem 8

Find the equation $y = mx + c$ of the line that best fits the data points $(0,4)$, $(1,-1)$, $(2,1)$, $(3,-3)$ and $(4,-1)$.

Solution

We are looking for the least square solution of the system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \\ 4 & 1 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} m \\ c \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} 4 \\ -1 \\ 1 \\ -3 \\ -1 \end{bmatrix}.$$

To find the solution we solve the system $A^T A\mathbf{x} = A^T\mathbf{b}$ which is

$$\begin{bmatrix} 30 & 10 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} m \\ c \end{bmatrix} \begin{bmatrix} -12 \\ 0 \end{bmatrix}.$$

Gauss elimination gives

$$\begin{bmatrix} 30 & 10 & -12 \\ 10 & 5 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -6 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -6 \\ 0 & 5 & 12 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -6/5 \\ 0 & 1 & 12/5 \end{bmatrix}.$$

The solution is $m = -6/5$ and $c = 12/5$. Thus the best fitting line is

$$y = -6/5x + 12/5.$$

Problem 9

Let A be the matrix $\begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix}$ and $\mathbf{u}$ be the vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

a) Verify that 2 is an eigenvalue of A and that $\mathbf{u}$ is an eigenvector of A (possibly with an eigenvalue different from 2).

b) Find all the eigenvalues of A and a basis for each eigenspace of A.

c) Is A orthogonally diagonalizable? If so, orthogonally diagonalize A.

Solution

a) To check that $\mathbf{u}$ is an eigenvector we just calculate

$$A\mathbf{u} = \begin{bmatrix} 3 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 1 & 3 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 5 \end{bmatrix} = 5\mathbf{u}.$$

Hence $\mathbf{u}$ is an eigenvector to the eigenvalue 5.

To verify that a is an eigenvalue of A we show via row reductions that $(A-2I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution:

$$A - 2I = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Hence not every column in $(A-2I)$ is a pivot column and $(A-2I)\mathbf{x} = \mathbf{0}$ has a nontrivial solution.

b) We have learned from a) that 5 and 2 are the eigenvalues of A. Moreover, every vector in $\mathbb{R}^3$ of the form

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

is an eigenvector for the eigenvalue 2. Hence the two vectors $\mathbf{v} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis of the eigenspace to eigenvalue 2. Since the dimensions of all eigenspaces have to add up to 3, we see that there are no other eigenvalues, and the eigenspace to eigenvalue 5 has dimension 1 with $\mathbf{u}$ as a basis vector.

c) It remains to orthonormalize the basis $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$. Since A is a symmetric matrix, we know that $\mathbf{u}$ is orthogonal to $\mathbf{v}$ and $\mathbf{w}$ (for $\mathbf{u}$ corresponds to a different eigenvalue). Hence we just need to normalize $\mathbf{u}$ and define a new basis vector

$$\mathbf{v}_1 := \frac{1}{||\mathbf{u}||}\mathbf{u} = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Next we orthogonalize $\mathbf{v}$ and $\mathbf{w}$ using the Gram-Schmidt process. We keep $\mathbf{v}$ and define a new vector $\tilde{\mathbf{w}}$ which is orthogonal to $\mathbf{v}$:

$$\tilde{\mathbf{w}} := \mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}.$$

It remains to normalize $\mathbf{v}$ and $\tilde{\mathbf{w}}$ to get the new vectors $\mathbf{v}_2$ and $\mathbf{v}_3$:

$$\mathbf{v}_2 := \frac{1}{||\mathbf{v}||}\mathbf{v} = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} \\ 1/\sqrt{2} \\ 0 \end{bmatrix}$$

and

$$\mathbf{v}_3 := \frac{1}{||\tilde{\mathbf{w}}||}\tilde{\mathbf{w}} = \sqrt{\frac{2}{3}} \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{6} \\ -1/\sqrt{6} \\ 2/\sqrt{6} \end{bmatrix}.$$

Hence we can orthogonally diagonalize A as $A = PDP^T$ with

$$D = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \; P = \begin{bmatrix} 1/\sqrt{3} & -1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 1/\sqrt{2} & -1/\sqrt{6} \\ 1/\sqrt{3} & 0 & 2/\sqrt{6} \end{bmatrix}, \text{ and } P^T = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ -1/\sqrt{2} & 1/\sqrt{2} & 0 \\ -1/\sqrt{6} & -1/\sqrt{6} & 2/\sqrt{6} \end{bmatrix}.$$

Problem 10

Let $W \subseteq \mathbb{R}^n$ be a subspace and $W^\perp$ be its orthogonal complement.

a) Show that $W^\perp$ is a subspace of $\mathbb{R}^n$.

b) Let $\mathbf{w}$ be a vector which lies both in W and in $W^\perp$ (i.e. $\mathbf{w} \in W \cap W^\perp$). Show that this implies $\mathbf{w} = \mathbf{0}$.

c) Let $\{\mathbf{w}_1, \dots, \mathbf{w}_r\}$ be a basis of W and let $\{\mathbf{v}_1, \dots, \mathbf{v}_s\}$ be a basis of $W^\perp$. Show that $\{\mathbf{w}_1, \dots, \mathbf{w}_r, \mathbf{v}_1, \dots, \mathbf{v}_s\}$ is a basis of $\mathbb{R}^n$.

Solution

a) Since the zero vector is orthogonal to every vector in $\mathbb{R}^n$, it is also an element in $W^\perp$. Let $\mathbf{u}$, $\mathbf{v}$ be arbitrary vectors in $W^\perp$, $\mathbf{w}$ be an arbitrary vector in W, and λ be any real number. Then we have:

$$(\mathbf{u}+\mathbf{v}) \cdot \mathbf{w} = \mathbf{u} \cdot \mathbf{w} + \mathbf{v} \cdot \mathbf{w} = 0 + 0 = 0, \text{ and } (\lambda \mathbf{u}) \cdot \mathbf{w} = \lambda(\mathbf{u} \cdot \mathbf{w}) = \lambda \cdot 0 = 0.$$

Hence $\mathbf{u}+\mathbf{v} \in W^\perp$ and $\lambda \mathbf{u} \in W^\perp$, and $W^\perp$ is indeed a subspace of $\mathbb{R}^n$.

b) Let $\mathbf{w}$ be a vector which lies both in W and $W^\perp$. Then $\mathbf{w} \in W^\perp$ implies that $\mathbf{w}$ is orthogonal to every vector in W and, in particular, $\mathbf{w}$ is orthogonal to itself. That means $\mathbf{w} \cdot \mathbf{w} = 0$, and hence $\mathbf{w}$ must be the zero vector in $\mathbb{R}^n$.

c) We need to show that $\{\mathbf{w}_1, \dots, \mathbf{w}_r, \mathbf{v}_1, \dots, \mathbf{v}_s\}$ is linearly independent and that it spans $\mathbb{R}^n$. We start with linear independence. Let $\lambda_1, \dots, \lambda_r, \mu_1, \dots, \mu_s$ be real numbers such that

$$\lambda_1 \mathbf{w}_1 + \ldots + \lambda_r \mathbf{w}_r + \mu_1 \mathbf{v}_1 + \ldots + \mu_s \mathbf{v}_s = \mathbf{0}.$$

This is equivalent to

$$\lambda_1 \mathbf{w}_1 + \ldots + \lambda_r \mathbf{w}_r = -(\mu_1 \mathbf{v}_1 + \ldots + \mu_s \mathbf{v}_s).$$

But the vector $\lambda_1 \mathbf{w}_1 + \ldots + \lambda_r \mathbf{w}_r$ is an element in W, whereas the vector $-(\mu_1 \mathbf{v}_1 + \ldots + \mu_s \mathbf{v}_s)$ is an element in $W^\perp$. By b), this implies

$$\lambda_1 \mathbf{w}_1 + \ldots + \lambda_r \mathbf{w}_r = \mathbf{0} = \mu_1 \mathbf{v}_1 + \ldots + \mu_s \mathbf{v}_s.$$

Since both sets $\{\mathbf{w}_1, \ldots, \mathbf{w}_r\}$ and $\{\mathbf{v}_1, \ldots, \mathbf{v}_s\}$ are linearly independent, this implies $\lambda_1 = \ldots = \lambda_r = 0$ and $\mu_1 = \ldots = \mu_s = 0$. This shows that $\{\mathbf{w}_1, \ldots, \mathbf{w}_r, \mathbf{v}_1, \ldots, \mathbf{v}_s\}$ is a linearly independent set of vectors.

It remains to show that $\{\mathbf{w}_1, \ldots, \mathbf{w}_r, \mathbf{v}_1, \ldots, \mathbf{v}_s\}$ spans $\mathbb{R}^n$. Let $\mathbf{y}$ be an arbitrary vector in $\mathbb{R}^n$. We learned that we can write $\mathbf{y}$ as a sum $\mathbf{y} = \mathbf{proj}_W\mathbf{y} + \mathbf{z}$ with $\mathbf{proj}_W\mathbf{y} \in W$ and $\mathbf{z} \in W^\perp$. By the assumptions, we can write $\mathbf{proj}_W\mathbf{y}$ as a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_r$ and $\mathbf{z}$ as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_s$. Hence we can also write $\mathbf{y}$ as a linear combination of $\mathbf{w}_1, \ldots, \mathbf{w}_r, \mathbf{v}_1, \ldots, \mathbf{v}_s$.