Problem 1  Find all complex numbers $z$ such that $z^3 = -8$ and plot them on the complex plane.

Solution We are looking for the answer in the polar form, $z = r e^{i\phi}$, the equation is $z^3 = r^3 e^{3i\phi} = 8 e^{i\pi}$. Thus $r^3 = 8, r = 2$ and $3\phi = \pi + 2\pi k$ where $k$ is integer. We get three distinct solutions taking $k = 0, 1, 2$ they have arguments $\phi_1 = \pi/3, \phi_2 = \pi$ and $\phi_3 = 5\pi/3 = -\pi/3$. Thus the solutions are

$$z_1 = 2e^{i\pi/3} = 1 + i\sqrt{3}, \quad z_2 = 2e^{i\pi} = -2, \quad z_3 = 2e^{-i\pi/3} = 1 - i\sqrt{3}$$

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Problem 2

a) Find two linearly independent solutions of the homogeneous differential equation

$$y'' + 2y' + 2y = 0$$

Solution First, we solve the characteristic equation $\lambda^2 + 2\lambda + 2 = 0$. It has two complex conjugate roots $\lambda_{1,2} = -1 \pm i$. Then two solutions are

$$y_1(t) = e^{-t} \cos t, \quad y_2(t) = e^{-t} \sin t$$

b) Find the steady-state solution of the differential equation

$$y'' + 2y' + 2y = 4 \cos 2t$$

Solution As we saw in part (a), solutions of the homogeneous equation tend to zero as $t \to \infty$. By the method of undetermined coefficients the steady-state solution
has the form $y_p(t) = A \cos 2t + B \sin 2t$. We compute the derivatives and plug them into the equation to determine the constants $A$ and $B$. We get

$$-4A \cos 2t - 4B \sin 2t - 4A \cos 2t + 4B \cos 2t + 2A \cos 2t + 2B \sin 2t = 4 \cos 2t$$

Comparing the coefficients for $\cos 2t$ and $\sin 2t$ on the right- and left-hand sides gives $4B - 2A = 4, 2B + 4A = 0$. Thus $A = -0.4, B = 0.8$. Finally, the answer is

$$y_p(t) = -0.4 \cos 2t + 0.8 \sin 2t$$

**Problem 3** Solve the initial value problem

$$y'' - 2y' - 8y = 12e^{-2t} + 8e^{2t}, \quad y(0) = 1, \quad y'(0) = 4$$

*Solution* First we find the general solution of the corresponding homogeneous equation $y'' - 2y' - 8y = 0$. The characteristic equation $\lambda^2 - 2\lambda - 8 = 0$ has two real roots $\lambda_1 = 4, \lambda_2 = -2$, then $y_h = c_1 e^{4t} + c_2 e^{-2t}$.

Next, we find a particular solution of the non-homogeneous equation, we can use the method of undetermined coefficients. Since $e^{-2t}$ solves the homogeneous equation we use the modification rule for this term and look for a solution of the form $y_p(t) = Ate^{-2t} + Be^{2t}$. Computing the derivatives and setting them in the equation gives

$$-4Ae^{-2t} + 4At e^{-2t} + 4Be^{2t} - 2Ae^{-2t} + 4At e^{-2t} - 4Be^{2t} - 8Ate^{-2t} - 8Be^{2t} = 12e^{-2t} + 8e^{2t}$$

We get $A = -2, B = -1$. Thus $y_p = -2te^{-2t} - e^{2t}$.

Finally, to solve the initial value problem we should specify the constants $c_1$ and $c_2$ in the general solution $y(t) = c_1 e^{4t} + c_2 e^{-2t} - 2te^{-2t} - e^{2t}$. We have $y'(t) = 4c_1 e^{4t} - 2c_2 e^{-2t} - 2e^{-2t} + 4te^{-2t} - 2e^{2t}$. The initial conditions give $1 = y(0) = c_1 + c_2 - 1$ and $4 = y'(0) = 4c_1 - 2c_2 - 4$. Solving for $c_1$ and $c_2$ we obtain $c_1 = 2, c_2 = 0$ and the answer is

$$y(t) = 2e^{4t} - 2te^{-2t} - e^{2t}$$
Problem 4  In this problem, we let $a_1$, $a_2$, $a_3$ and $b$ be the following vectors in $\mathbb{R}^4$:

$$
\begin{align*}
   a_1 &= \begin{bmatrix} 1 \\ 4 \\ 1 \\ 2 \end{bmatrix}, \\
   a_2 &= \begin{bmatrix} 3 \\ 13 \\ 1 \\ 9 \end{bmatrix}, \\
   a_3 &= \begin{bmatrix} 9 \\ 31 \\ 5 \\ 29 \end{bmatrix}, \\
   b &= \begin{bmatrix} 4 \\ -17 \\ 14 \\ 13 \end{bmatrix}
\end{align*}
$$

a) Show that the vector $b$ is a linear combination of $a_1$, $a_2$ and $a_3$.

**Solution** To show that $b$ is a linear combination of $a_1$, $a_2$ and $a_3$, we find real numbers $c_1$, $c_2$ and $c_3$ such that $c_1a_1 + c_2a_2 + c_3a_3 = b$. It can be done by solving the linear system of equations with the augmented matrix

$$
\begin{bmatrix}
   1 & 3 & 9 & | & 4 \\
   4 & 13 & 31 & | & -17 \\
   1 & 1 & 5 & | & 14 \\
   2 & 9 & 29 & | & 13
\end{bmatrix}
$$

We perform the Gauss elimination

$$
\begin{bmatrix}
   1 & 3 & 9 & 4 \\
   4 & 13 & 31 & -17 \\
   1 & 1 & 5 & 14 \\
   2 & 9 & 29 & 13
\end{bmatrix} \rightarrow
\begin{bmatrix}
   1 & 3 & 9 & 4 \\
   0 & 1 & -5 & -33 \\
   0 & -2 & -4 & 10 \\
   0 & 3 & 11 & 5
\end{bmatrix} \rightarrow
\begin{bmatrix}
   1 & 3 & 9 & 4 \\
   0 & 1 & -5 & -33 \\
   0 & 0 & -14 & -56 \\
   0 & 0 & 26 & 104
\end{bmatrix} \rightarrow
\begin{bmatrix}
   1 & 3 & 9 & 4 \\
   0 & 1 & -5 & -33 \\
   0 & 0 & 0 & 0 \\
   0 & 0 & 0 & 0
\end{bmatrix}
$$

Now the back substitution gives $c_3 = 4, c_2 = -13, c_1 = 7$. Therefore $b = 7a_1 - 13a_2 + 4a_3$.

b) Let $T: \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be a linear transformation such that

$$
T(a_1) = \begin{bmatrix} 8 \\ 0 \end{bmatrix}, \quad T(a_2) = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{and} \quad T(a_3) = \begin{bmatrix} -5 \\ 5 \end{bmatrix}.
$$

Find $T(b)$.

**Solution** We have from part (a) that $b = 7a_1 - 13a_2 + 4a_3$. Since $T$ is a linear transformation it implies $T(b) = 7T(a_1) - 13T(a_2) + 4T(a_3)$, thus

$$
T(b) = 7 \begin{bmatrix} 8 \\ 0 \end{bmatrix} - 13 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + 4 \begin{bmatrix} -5 \\ 5 \end{bmatrix} = \begin{bmatrix} 10 \\ 7 \end{bmatrix}
$$
Problem 5

Set

\[ M = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 7 \\ 1 & 2 & 2 & 5 \\ 3 & 6 & 6 & 15 \end{bmatrix} \]

a) Find a basis for the row space and a basis for the column space of \( M \). What is the rank of \( M \)?

**Solution** First we perform the Gauss elimination on \( M \)

\[
\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 7 \\ 1 & 2 & 2 & 5 \\ 3 & 6 & 6 & 15 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{bmatrix}
\]

Then a basis for the row space of \( M \) is \( \{(1,2,1,3),(0,0,1,3)\} \). A basis for the column space of \( M \) can be obtained by taking the first and third columns of \( M \), that is \( \{(1,2,1,3)^T,(1,3,2,6)^T\} \). The rank of \( M \) equals the dimension of the row and column space, which is the number of vectors in a basis, thus \( \text{rank}(M) = 2 \).

b) Find an orthogonal basis for the null space of \( M \).

**Solution** Using the Gauss elimination from part (a) we see that \( \text{Null}(M) = \text{null} \left( \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix} \right) \). Here we have two free variables \( x_2 \) and \( x_4 \) and the other two are given by \( x_3 = -3x_4 \) and \( x_1 = -2x_2 - x_3 - 2x_4 = -2x_2 + x_4 \), thus the general solution of the equation \( Mx = 0 \) is

\[
x = s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}
\]

Thus a basis for the solution space is \( \{\mathbf{v}_1 = (-2,1,0,0), \mathbf{v}_2 = (1,0,-3,1)\} \). It is not orthogonal, \( \mathbf{v}_1 \cdot \mathbf{v}_2 = -2 \). To find an orthogonal basis we apply the Gram-Schmidt algorithm to those two vectors and obtain

\[
\mathbf{u}_1 = \mathbf{v}_1 = (-2,1,0,0),
\]

\[
\mathbf{u}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1} \mathbf{v}_1 = (1,0,-3,1) + 0.4(-2,1,0,0) = (0.2,0.4,-3,1)
\]

And an orthogonal basis for the null space of \( M \) is \( \{(-2,1,0,0), (0.2,0.4,-3,1)\} \)
Problem 6  Let $A = \begin{bmatrix} -3 & 3 \\ 2 & -4 \end{bmatrix}$

a) Find the eigenvalues and corresponding eigenvectors of $A$.

Solution To find the eigenvalues we compute the characteristic polynomial, $\det(A - \lambda I) = (-3 - \lambda)(-4 - \lambda) - 6 = 6 + 7\lambda + \lambda^2$. The roots are $\lambda_1 = -1$ and $\lambda_2 = -6$.

First, consider $\lambda_1 = -1$, we have $A + I = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$ a corresponding eigenvector is $v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Next, for $\lambda_2 = -6$ we get $A + 6I = \begin{bmatrix} 6 & 6 \\ 2 & 2 \end{bmatrix}$ and a second eigenvector $v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Note that there is some freedom in choosing eigenvectors, each of them can be multiplied by a non-zero constant. We get follow answer:

$$\lambda_1 = -1, \; \lambda_2 = -6, \; v_1 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}, \; v_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

b) Solve the initial value problem

$$x'(t) = Ax, \; x(0) = \begin{bmatrix} 150 \\ 50 \end{bmatrix}$$

and find $x(5)$.

Solution The general solution of the system of ODEs $x' = Ax$ is $x(t) = c_1e^{-t}v_1 + c_2e^{-6t}v_2$ where $v_1$ and $v_2$ are eigenvectors corresponding to eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$, which were computed in part (a).

We use the initial condition $x(0) = \begin{bmatrix} 150 \\ 50 \end{bmatrix}$ to specify the constants $c_1$ and $c_2$. The equation is

$$c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 150 \\ 50 \end{bmatrix}$$

Then $c_1 = 40$ and $c_2 = -30$. The solution to the initial value problem is

$$x(t) = 40e^{-t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 30e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 120e^{-t} + 30e^{-6t} \\ 80e^{-t} - 30e^{-6t} \end{bmatrix}$$
Plugging in $t = 5$ we obtain

$$x(5) \approx \begin{bmatrix} 0.81 \\ 0.54 \end{bmatrix}$$

**Problem 7** A symmetric $3 \times 3$ matrix $A$ has eigenvalues 1, 2 and 3.

a) Prove that the matrix $A^3 - A + I$ is diagonalizable.

**Solution 1** For this part it is not important that $A$ is symmetric. We know that $A$ is a $3 \times 3$ matrix with three distinct real eigenvalues, it means that there exists a basis for $\mathbb{R}^3$ of eigenvectors of $A$ and $A$ is diagonalizable. We can write $A = PDP^{-1}$, where $D$ is diagonal. Then $A^3 = PD^3P^{-1}$ and

$$A^3 - A + I = PD^3P^{-1} - PDP^{-1} + PP^{-1} = P(D^3 - D + I)P^{-1}$$

This is a diagonalization of matrix $A$ since $D_1 = D^3 - D + I$ is a diagonal matrix.

We actually know more,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad D_1 = D^3 - D + I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

**Solution 2** Note that if $v$ is an eigenvector of $A$, $Av = \lambda v$ then it is also an eigenvector of $A^3 - A + I$ since $(A^3 - A + I)v = (\lambda^3 - \lambda + 1)v$. We know that $A$ has three distinct eigenvalues and thus three linearly independent eigenvectors. Then $A^3 - A + I$ has the same three linearly independent eigenvectors and it is diagonalizable.

b) Suppose that $v_1 = [1 \ 1 \ 2]^T$ and $v_2 = [1 \ 1 \ -1]^T$ are eigenvectors corresponding to eigenvalues 1 and 2. Find an eigenvector that corresponds to the eigenvalue 3.

**Solution** Now we will use the fact that $A$ is symmetric. It implies that eigenvectors corresponding to distinct eigenvalues are pairwise orthogonal. Then our problem is to find a vector $v_3$ in $\mathbb{R}^3$ which is orthogonal to two given vectors $v_1$ and $v_2$. There are different ways to do it, one is to write
\[ \mathbf{v}_3 = \begin{bmatrix} x \\ y \\ z \end{bmatrix} \] and rewrite the orthogonality conditions as linear equations on \( x, y, z \). We are looking for a non-trivial solution of the system

\begin{align*}
x + y + 2z &= 0 \\
x + y - z &= 0
\end{align*}

One non-zero solution is \( x = 1, y = -1, z = 0 \) and all other are actually multiples of that one. Thus we get \( \mathbf{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} \).