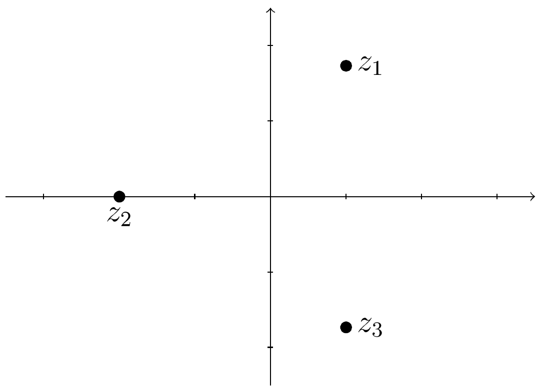
Problem 1 Find all complex numbers z such that $z^3 = -8$ and plot them on the complex plane.

Solution We are looking for the answer in the polar form, $z = re^{i\phi}$, the equation is $z^3 = r^3 e^{3i\phi} = 8e^{i\pi}$. Thus $r^3 = 8, r = 2$ and $3\phi = \pi + 2\pi k$ where k is integer. We get three distinct solutions taking $k = 0, 1, 2$ they have arguments $\phi_1 = \pi/3$, $\phi_2 = \pi$ and $\phi_3 = 5\pi/3 = -\pi/3$. Thus the solutions are

$$\underline{z_1 = 2e^{i\pi/3} = 1 + i\sqrt{3}, \quad z_2 = 2e^{i\pi} = -2, \quad z_3 = 2e^{-i\pi/3} = 1 - i\sqrt{3}}$$



Problem 2

a) Find two linearly independent solutions of the homogeneous differential equation

$$y'' + 2y' + 2y = 0$$

Solution First, we solve the characteristic equation $\lambda^2 + 2\lambda + 2 = 0$. It has two complex conjugate roots $\lambda_{1,2} = -1 \pm i$. Then two solutions are

$$\underline{y_1(t) = e^{-t}\cos t, \quad y_2(t) = e^{-t}\sin t}$$

b) Find the steady-state solution of the differential equation

$$y'' + 2y' + 2y = 4\cos 2t$$

Solution As we saw in part (a), solutions of the homogeneous equation tend to zero as $t \to \infty$. By the method of undetermined coefficients the steady-state solution has the form $y_p(t) = A\cos 2t + B\sin 2t$. We compute the derivatives and plug them into the equation to determine the constants A and B. We get

$$-4A\cos 2t - 4B\sin 2t - 4A\sin 2t + 4B\cos 2t + 2A\cos 2t + 2B\sin 2t = 4\cos 2t$$

Comparing the coefficients for $\cos 2t$ and $\sin 2t$ on the right- and left-hand sides gives $4B - 2A = 4,\ 2B + 4A = 0$. Thus $A = -0.4,\ B = 0.8$. Finally, the answer is

$$\underline{y_p(t) = -0.4\cos 2t + 0.8\sin 2t}$$

Problem 3 Solve the initial value problem

$$y'' - 2y' - 8y = 12e^{-2t} + 8e^{2t}, \quad y(0) = 1,\ y'(0) = 4$$

Solution First we find the general solution of the corresponding homogeneous equation $y'' - 2y' - 8y = 0$. The characteristic equation $\lambda^2 - 2\lambda - 8 = 0$ has two real roots $\lambda_1 = 4,\ \lambda_2 = -2$, then $y_h = c_1e^{4t} + c_2e^{-2t}$.

Next, we find a particular solution of the non-homogeneous equation, we can use the method of undetermined coefficients. Since e^{-2t} solves the homogeneous equation we use the modification rule for this term and look for a solution of the form $y_p(t) = Ate^{-2t} + Be^{2t}$. Computing the derivatives and setting them in the equation gives

$$-4Ae^{-2t} + 4Ate^{-2t} + 4Be^{2t} - 2Ae^{-2t} + 4Ate^{-2t} - 4Be^{2t} - 8Ate^{-2t} - 8Be^{2t} = 12e^{-2t} + 8e^{2t}$$

We get $A = -2,\ B = -1$. Thus $y_p = -2te^{-2t} - e^{2t}$.

Finally, to solve the initial value problem we should specify the constants c_1 and c_2 in the general solution $y(t) = c_1e^{4t} + c_2e^{-2t} - 2te^{-2t} - e^{2t}$. We have $y'(t) = 4c_1e^{4t} - 2c_2e^{-2t} - 2e^{-2t} + 4te^{-2t} - 2e^{2t}$. The initial conditions give $1 = y(0) = c_1 + c_2 - 1$ and $4 = y'(0) = 4c_1 - 2c_2 - 4$. Solving for c_1 and c_2 we obtain $c_1 = 2, c_2 = 0$ and the answer is

$$\underline{y(t) = 2e^{4t} - 2te^{-2t} - e^{2t}}$$

Problem 4 In this problem, we let $\mathbf{a_1}$, $\mathbf{a_2}$, $\mathbf{a_3}$ and $\mathbf{b}$ be the following vectors in $\mathbb{R}^4$:

$$\mathbf{a_1} = \begin{bmatrix}1\\4\\1\\2\end{bmatrix} \qquad \mathbf{a_2} = \begin{bmatrix}3\\13\\1\\9\end{bmatrix} \qquad \mathbf{a_3} = \begin{bmatrix}9\\31\\5\\29\end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix}4\\-17\\14\\13\end{bmatrix}$$

a) Show that the vector $\mathbf{b}$ is a linear combination of $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$.

Solution To show that $\mathbf{b}$ is a linear combination of $\mathbf{a_1}$, $\mathbf{a_2}$ and $\mathbf{a_3}$, we find real numbers c_1, c_2 and c_3 such that $c_1\mathbf{a_1} + c_2\mathbf{a_2} + c_3\mathbf{a_3} = \mathbf{b}$. It can be done by solving the linear system of equations with the augmented matrix

$$\left[\begin{array}{ccc|c}1 & 3 & 9 & 4\\4 & 13 & 31 & -17\\1 & 1 & 5 & 14\\2 & 9 & 29 & 13\end{array}\right]$$

We perform the Gauss elimination

$$\begin{bmatrix}1 & 3 & 9 & 4\\4 & 13 & 31 & -17\\1 & 1 & 5 & 14\\2 & 9 & 29 & 13\end{bmatrix} \to \begin{bmatrix}1 & 3 & 9 & 4\\0 & 1 & -5 & -33\\0 & -2 & -4 & 10\\0 & 3 & 11 & 5\end{bmatrix} \to \begin{bmatrix}1 & 3 & 9 & 4\\0 & 1 & -5 & -33\\0 & 0 & -14 & -56\\0 & 0 & 26 & 104\end{bmatrix} \to \begin{bmatrix}1 & 3 & 9 & 4\\0 & 1 & -5 & -33\\0 & 0 & -14 & -56\\0 & 0 & 0 & 0\end{bmatrix}$$

Now the back substitution gives $c_3 = 4, c_2 = -13, c_1 = 7$. Therefore $\underline{\mathbf{b} = 7\mathbf{a_1} - 13\mathbf{a_2} + 4\mathbf{a_3}}$.

b) Let $T\colon \mathbb{R}^4 \to \mathbb{R}^2$ be a linear transformation such that

$$T(\mathbf{a_1}) = \begin{bmatrix}8\\0\end{bmatrix}, \qquad T(\mathbf{a_2}) = \begin{bmatrix}2\\1\end{bmatrix} \qquad \text{and} \qquad T(\mathbf{a_3}) = \begin{bmatrix}-5\\5\end{bmatrix}.$$

Find $T(\mathbf{b})$.

Solution We have from part (a) that $\mathbf{b} = 7\mathbf{a_1} - 13\mathbf{a_2} + 4\mathbf{a_3}$. Since T is a linear transformation it implies $T(\mathbf{b}) = 7T(\mathbf{a_1}) - 13T(\mathbf{a_2}) + 4T(\mathbf{a_3})$, thus

$$\underline{T(\mathbf{b})} = 7\begin{bmatrix}8\\0\end{bmatrix} - 13\begin{bmatrix}2\\1\end{bmatrix} + 4\begin{bmatrix}-5\\5\end{bmatrix} = \underline{\begin{bmatrix}10\\7\end{bmatrix}}$$

Problem 5 Set

$$M = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 7 \\ 1 & 2 & 2 & 5 \\ 3 & 6 & 6 & 15 \end{bmatrix}$$

a) Find a basis for the row space and a basis for the column space of M. What is the rank of M?

Solution First we perform the Gauss elimination on M

$$\begin{bmatrix} 1 & 2 & 1 & 2 \\ 2 & 4 & 3 & 7 \\ 1 & 2 & 2 & 5 \\ 3 & 6 & 6 & 15 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 3 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Then a basis for the row space of M is $\underline{\{(1,2,1,2),(0,0,1,3)\}}$. A basis for the column space of M can be obtained by taking the first and third columns of M, that is $\underline{\{(1,2,1,3)^T,(1,3,2,6)^T\}}$. The rank of M equals the dimension of the row and column space, which is the number of vectors in a basis, thus $\underline{\text{rank}(M) = 2}$.

b) Find an orthogonal basis for the null space of M.

Solution Using the Gauss elimination from part (a) we see that $\text{Null}(M) = \text{null}\left(\begin{bmatrix} 1 & 2 & 1 & 2 \\ 0 & 0 & 1 & 3 \end{bmatrix}\right)$. Here we have two free variables x_2 and x_4 and the other two are given by $x_3 = -3x_4$ and $x_1 = -2x_2 - x_3 - 2x_4 = -2x_2 + x_4$, thus the general solution of the equation $M\mathbf{x} = \mathbf{0}$ is

$$\mathbf{x} = s\begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ 0 \\ -3 \\ 1 \end{bmatrix}$$

Thus a basis for the solution space is $\{\mathbf{v_1} = (-2,1,0,0), \mathbf{v_2} = (1,0,-3,1)\}$. It is not orthogonal, $\mathbf{v_1} \cdot \mathbf{v_2} = -2$. To find an orthogonal basis we apply the Gram-Schmidt algorithm to those two vectors and obtain

$$\mathbf{u_1} = \mathbf{v_1} = (-2,1,0,0),$$
$$\mathbf{u_2} = \mathbf{v_2} - \frac{\mathbf{v_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\mathbf{v_1} = (1,0,-3,1) + 0.4(-2,1,0,0) = (0.2,0.4,-3,1)$$

And an orthogonal basis for the null space of M is $\underline{\{(-2,1,0,0),\ (0.2,0.4,-3,1)\}}$

Problem 6 Let $A = \begin{bmatrix} -3 & 3 \\ 2 & -4 \end{bmatrix}$

a) Find the eigenvalues and corresponding eigenvectors of A.

Solution To find the eigenvalues we compute the characteristic polynomial, $\det(A - \lambda I) = (-3 - \lambda)(-4 - \lambda) - 6 = 6 + 7\lambda + \lambda^2$. The roots are $\lambda_1 = -1$ and $\lambda_2 = -6$.

First, consider $\lambda_1 = -1$, we have $A + I = \begin{bmatrix} -2 & 3 \\ 2 & -3 \end{bmatrix}$ a corresponding eigenvector is $\mathbf{v_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$.

Next, for $\lambda_2 = -6$ we get $A + 6I = \begin{bmatrix} 6 & 6 \\ 2 & 2 \end{bmatrix}$ and a second eigenvector $\mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Note that there is some freedom in choosing eigenvectors, each of them can be multiplied by a non-zero constant. We get follow answer:

$$\underline{\lambda_1 = -1,\ \lambda_2 = -6, \quad \mathbf{v_1} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},\ \mathbf{v_2} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}}$$

b) Solve the initial value problem

$$\mathbf{x}'(t) = A\mathbf{x}, \quad \mathbf{x}(0) = \begin{bmatrix} 150 \\ 50 \end{bmatrix}$$

and find $\mathbf{x}(5)$.

Solution The general solution of the system of ODEs $\mathbf{x}' = A\mathbf{x}$ is $\mathbf{x}(t) = c_1 e^{-t}\mathbf{v_1} + c_2 e^{-6t}\mathbf{v_2}$ where $\mathbf{v_1}$ and $\mathbf{v_2}$ are eigenvectors corresponding to eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -6$, which were computed in part (a).

We use the initial condition $[\mathbf{x}(0) = \begin{bmatrix} 150 \\ 50 \end{bmatrix}$ to specify the constants c_1 and c_2. The equation is

$$c_1 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + c_2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 150 \\ 50 \end{bmatrix}$$

Then $c_1 = 40$ and $c_2 = -30$. The solution to the initial value problem is

$$\underline{\mathbf{x}(t) = 40e^{-t} \begin{bmatrix} 3 \\ 2 \end{bmatrix} - 30e^{-6t} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 120e^{-t} + 30e^{-6t} \\ 80e^{-t} - 30e^{-6t} \end{bmatrix}}$$

Plugging in $t = 5$ we obtain

$$\mathbf{x}(5) \approx \begin{bmatrix} 0.81 \\ 0.54 \end{bmatrix}$$

Problem 7 A symmetric 3×3 matrix A has eigenvalues $1, 2$ and 3.

a) Prove that the matrix $A^3 - A + I$ is diagonalizable.

Solution 1 Foe this part it is not important that A is symmetric. We know that A is a 3×3 matrix with three distinct real eigenvalues, it means that there exists a basis for $\mathbb{R}^3$ of eigenvectors of A and A is diagonalizable. We can write $A = PDP^{-1}$, where D is diagonal . Then $A^3 = PD^3P^{-1}$ and

$$A^3 - A + I = PD^3P^{-1} - PDP^{-1} + PP^{-1} = P(D^3 - D + I)P^{-1}$$

This is a diagonalization of matrix A since $D_1 = D^3 - D + I$ is a diagonal matrix.

We actually know more,

$$D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{bmatrix}, \quad D_1 = D^3 - D + I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 7 & 0 \\ 0 & 0 & 25 \end{bmatrix}$$

Solution 2 Note that if $\mathbf{v}$ is an eigenvector of A, $A\mathbf{v} = \lambda\mathbf{v}$ then it is also an eiganvector of $A^3 - A + I$ since $(A^3 - A + I)\mathbf{v} = (\lambda^3 - \lambda + 1)\mathbf{v}$. We know that A has three disctinct eigenvalues and thus three linearly independent eigenvectors. Then $A^3 - A + I$ has the same three linearly independent eigenvectors and it is diagonalizable.

b) Suppose that $\mathbf{v}_1 = [1\ 1\ 2]^T$ and $\mathbf{v}_2 = [1\ 1\ -1]^T$ are eigenvectors corresponding to eigenvalues 1 and 2. Find an eigenvector that corresponds to the eigenvalue 3.

Solution Now we will use the fact that A is symmetric. It implies that eigenvectors corresponding to distinct eigenvalues are pairwise orthogonal. Then our problem is to find a vector $\mathbf{v_3}$ in $\mathbb{R}^3$ which is orthogonal to two given vectors $\mathbf{v_1}$ and $\mathbf{v_2}$. There are different ways to do it, one is to write

$\mathbf{v_3} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ and rewrite the orthogonality conditions as linear equations on x, y, z. We are looking for a non-trivial solution of the system

$$\begin{aligned} x + y + 2z &= 0 \\ x + y - z &= 0 \end{aligned}$$

One non-zero solution is $x = 1, y = -1, z = 0$ and all other are actually multiples of that one. Thus we get $\underline{\mathbf{v_3} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}}$.