Department of Mathematical Sciences

# Examination paper for **TMA4110 Matematikk 3**

**Academic contact during examination:** Eugenia Malinnikova
**Phone:** 73550257

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**Problem 1** Solve the quadratic equation $z^2 + (4+2i)z + 3 = 0$, write the solutions in normal form.

*Solution* We use the formula for solutions of the quadratic equation

$$z_{1,2} = -(2+i) \pm \sqrt{(2+i)^2 - 3} = -2 - i \pm \sqrt{4i} = -2 - i + 2\sqrt{i}.$$

Now we have $i = e^{i\pi/2}$ and then one of the roots is given by $\sqrt{i} = e^{i\pi/4} = \sqrt{2}/2 + \sqrt{2}/2i$. Thus

$$\underline{z_1 = -2 + \sqrt{2} + (-1 + \sqrt{2})i}, \quad \underline{z_2 = -2 - \sqrt{2} + (-1 - \sqrt{2})i}$$

**Problem 2**

**a)** Solve the initial value problem

$$x'' + 6x' + 8x = 0, \quad x(0) = 0, \; x'(0) = 8.$$

What is the maximal value attained by this solution $x(t)$ for $t > 0$?

*Solution* First we find the roots of the characteristic equation $\lambda^2 + 6\lambda + 8 = 0$, we have $\lambda_1 = -2$ and $\lambda_2 = -4$. Then the general solution to the homogeneous equation is $x(t) = c_1 e^{-2t} + c_2 e^{-4t}$. To find the constants, we use initial conditions, clearly, $x(0) = c_1 + c_2$ and $x'(0) = -2c_1 - 4c_2$. Solving the system, $c_1 + c_2 = 0, -2c_1 - 4c_2 = 8$ we obtain $c_1 = 4$, $c_2 = -4$. Thus $\underline{x(t) = 4e^{-2t} - 4e^{-4t}}$.

To find the maximum value, we compute $x'(t) = -8e^{-2t} + 16e^{-4t}$, if $e^{2t_0} = 2$ then $x'$ is positive on $(0, t_0)$ and negative on $(t_0, +\infty)$. Therefore the maximum value of $x$ on $(0, +\infty)$ is attained at $t_0$ and $\underline{x(t_0)} = 4e^{-2t_0} - 4e^{-4t_0} = 2 - 1 = \underline{1}$.

**b)** Find the steady-state solution of the equation

$$x'' + 6x' + 8x = 4\cos 2t.$$

*Solution* We consider the complex equation $z'' + 6z' + 8z = 4e^{2it}$, such that the real part of a solution is a solution of our initial equation. We are looking for a solution of the form $z(t) = ae^{2it}$. We have

$$z'' + 6z' + 8z = ((2i)^2 + 6(2i) + 8)ae^{2it} = P(2i)z(t), \; P(w) = w^2 + 6w + 8.$$

Hence $z(t) = 4e^{2it}/P(2i)$ and $1/P(2i) = (4 + 12i)^{-1} = (1 - 3i)/40$ and

$$z(t) = (0.1 - 0.3i)e^{2it} = 0.1\cos 2t + 0.3\sin 2t + i(0.1\sin 2t - 0.3\cos 2t),$$

$$\underline{x(t) = 0.1 \cos 2t + 0.3 \sin 2t.}$$

*Alternative solution* We use method of undetermined coefficients to find the particular solution of the form $x(t) = a\cos 2t + b\sin 2t$. The derivatives are: $x'(t) = -2a\sin 2t + 2b\cos 2t$ and $x''(t) = -4a\cos 2t - 4b\sin 2t$. Then

$$\begin{aligned} x''+6x+8x = -4a\cos 2t - 4b\sin 2t + 6(-2a\sin 2t + 2b\cos 2t) + 8(a\cos 2t + \sin 2t) \\ = (4a + 12b)\cos 2t + (4b - 12a)\sin 2t. \end{aligned}$$

We are solving the equation $x'' + 6x' + 8x = 4\cos 2t$. Therefore we want to find $a, b$ such that $4a + 12b = 4$ and $4b - 12a = 0$, we get $a = 0.1$, $b = 0.3$, the answer is

$$\underline{x(t) = 0.1 \cos 2t + 0.3 \sin 2t.}$$

**Problem 3** Find general solution of the equation

$$y'' + y = 3x + \tan(x).$$

(Hint $\int(\cos x)^{-1}dx = \ln|\sec x + \tan x|$.)

*Solution* First, solutions of the corresponding homogeneous equation are $y_1(x) = \cos x$ and $y_2(x) = \sin x$.

To find the solution of the non-homogeneous equation we divide the right hand-side into two parts, $r_1(x) = 3x$ and $r_2(x) = \tan x$. We can use the method of undetermined coefficients for the first part, we look for $y_{p_1} = ax + b$, then $y'' = 0$ and $y'' + y = ax + b$, thus $y_{p_1} = 3x$ solves the equation $y'' + y = 3x$.

For the second equation we use the method of variation of parameters,

$$W(y_1, y_2) = y_1 y_2' - y_1' y_2 = \cos^2 x + \sin^2 x = 1.$$

Thus $y_{p_2} = -y_1 \int y_2 \tan(x)dx + y_2 \int y_1 \tan(x)dx$,

$$\begin{aligned} y_{p_2}(x) = -\cos x \int \frac{\sin^2 x}{\cos x}dx + \sin x \int \sin x dx = \\ -\cos x \int (\cos x)^{-1}dx + \cos x \int \cos x dx + \sin x \int \sin x dx = \\ -\cos x \ln|\sec x + \tan x| + \cos x \sin x - \sin x \cos x = -\cos x \ln|\sec x + \tan x| \end{aligned}$$

Summing up the terms, we get the general solution

$$\underline{y(t) = y_h(t) + y_{p_1}(t) + y_{p_2}(t) = c_1 \cos x + c_2 \sin x + 3x - \cos x \ln|\sec x + \tan x|}$$

**Problem 4** Let

$$A = \begin{bmatrix} 1 & t \\ t & 2 \end{bmatrix}.$$

**a)** For which values of $t$ does the equation $A\mathbf{x} = \mathbf{b}$ have a solution for any $\mathbf{b}$ in $\mathbb{R}^2$?

*Solution* The equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ if and only if the matrix $A$ is invertible. This happens if and only if $\det(A) \neq 0$. We have $\det(A) = 2 - t^2$. Thus the equation always has a solution when $\underline{t \neq \pm\sqrt{2}}$.

*Alternatively*, we can start by performing Gauss elimination:

$$\begin{bmatrix} 1 & t \\ t & 2 \end{bmatrix} \xrightarrow{R_2 - tR_1} \begin{bmatrix} 1 & t \\ 0 & 2 - t^2 \end{bmatrix}$$

There is no zero-rows in the final matrix if and only if $t \neq \sqrt{2}$. Thus the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all $\mathbf{b}$ if and only if $\underline{t \neq \pm\sqrt{2}}$

**b)** Find an LU decomposition of $A$ (the result will depend on the parameter $t$).

*Solution* We apply Gauss elimination (see above) and get $U = \begin{bmatrix} 1 & t \\ 0 & 2 - t^2 \end{bmatrix}$, to get $L$ we remind that the only row operation in the Gauss elimination was adding $(-t)$ times the first row to the second one. Thus $L = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix}$, a simple computation confirms that

$$\underline{A = \begin{bmatrix} 1 & 0 \\ t & 1 \end{bmatrix} \begin{bmatrix} 1 & t \\ 0 & 2 - t^2 \end{bmatrix}.}$$

**Problem 5** Given the following vectors in $\mathbb{R}^4$

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 4 \\ -3 \\ -2 \\ 4 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 3 \\ -2 \\ 1 \\ 1 \end{pmatrix},$$

let $V = \operatorname{Span}\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

**a)** Are the vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ linearly independent? Find a basis for $V$.

*Solutuon* We consider the matrix with column vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ and $\mathbf{v_4}$ and perform the Gauss elimination to find out if the columns are linearly dependent or not.

$$C = \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 0 & -3 & -2 \\ 2 & 1 & -2 & 1 \\ 0 & -1 & 4 & 1 \end{bmatrix} \xrightarrow{R_3-2R_1} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 0 & -3 & -2 \\ 0 & 1 & -10 & -5 \\ 0 & -1 & 4 & 1 \end{bmatrix} \xrightarrow{SWAP(R_2,R_3)}$$
$$\begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -10 & -5 \\ 0 & 0 & -3 & -2 \\ 0 & -1 & 4 & 1 \end{bmatrix} \xrightarrow{R_4+R_2} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -10 & -5 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & -6 & -4 \end{bmatrix} \xrightarrow{R_4-2R_3} \begin{bmatrix} 1 & 0 & 4 & 3 \\ 0 & 1 & -10 & -5 \\ 0 & 0 & -3 & -2 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Since there is no pivot element in the last column, the columns are linearly dependent. There are pivot elements in the first three columns, thus $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ form a basis for $V$.

**b)** Find an orthogonal basis for $V$.

*Solution* We use the Gram-Schmidt algorithm to find an orthogonal basis:

$$\mathbf{u}_1 = \mathbf{v}_1 = \begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix},$$

$$\tilde{\mathbf{u}}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ -1 \end{pmatrix} - \frac{2}{5}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} = \frac{1}{5}\begin{pmatrix} -2 \\ 0 \\ 1 \\ -5 \end{pmatrix}, \quad \mathbf{u}_2 = \begin{pmatrix} -2 \\ 0 \\ 1 \\ -5 \end{pmatrix}$$

$$\mathbf{u}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\mathbf{u}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\mathbf{u}_2 = \begin{pmatrix} 4 \\ -3 \\ -2 \\ 4 \end{pmatrix} - \frac{0}{5}\begin{pmatrix} 1 \\ 0 \\ 2 \\ 0 \end{pmatrix} - \frac{-30}{30}\begin{pmatrix} -2 \\ 0 \\ 1 \\ -5 \end{pmatrix} = \begin{pmatrix} 2 \\ -3 \\ -1 \\ -1 \end{pmatrix}$$

**c)** Does there exist a vector $\mathbf{u} \neq \mathbf{0}$ in $\mathbb{R}^4$ which is orthogonal to $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4$?

Yes, since according to part (a) the dimension of $V$ is three and $V$ is a subspace of $\mathbb{R}^4$, there exists a non-zero vector $\mathbf{u}$ in $V^\perp$. One can find such vector by solving the system $C^T\mathbf{u} = 0$;for example $\mathbf{u} = [-2 \ -2 \ 1 \ 1]^T$ is such a vector.

**Problem 6**

**a)** Find (complex) eigenvalues and (complex) eigenvectors of the matrix

$$\begin{bmatrix} 1 & -2 \\ 1 & 3 \end{bmatrix}$$

*Solution* First, the characteristic polynomial is $\det(B-\lambda I) = (1-\lambda)(3-\lambda)+2 = \lambda^2 - 4\lambda + 5 = (\lambda-2)^2+1$. The roots are $\underline{\lambda_1 = 2+i}$, $\underline{\lambda_2 = \overline{\lambda_1} = 2-i}$. We find an eigenvector corresponding to $\lambda_1$,

$$B - (2+i)i = \begin{bmatrix} -1-i & -2 \\ 1 & 1-i \end{bmatrix}, \quad \underline{\mathbf{v}_1 = \begin{bmatrix} 2 \\ -1-i \end{bmatrix}}$$

For $\lambda_2 = \overline{\lambda_1}$, we obtain $\underline{\mathbf{v}_2 = \overline{\mathbf{v}_1} = \begin{bmatrix} 2 \\ -1+i \end{bmatrix}}$.

**b)** Find the solution of the system

$$\begin{aligned} x_1' &= x_1 - 2x_2 \\ x_2' &= x_1 + 3x_2 \end{aligned}$$

that satisfies the initial conditions $x_1(0) = 1$ and $x_2(0) = 1$. Write down the answer using real-valued functions.

*Solution* We know that $\mathbf{v}_1 e^{\lambda_1 t}$ and $\mathbf{v}_2 e^{\lambda_2 t}$ are (complex conjugate) solutions of this system, we find two real solutions:

$$\frac{1}{2}\left(\begin{bmatrix} 2 \\ -1-i \end{bmatrix} e^{(2+i)t} + \begin{bmatrix} 2 \\ -1+i \end{bmatrix} e^{(2-i)t}\right)$$
$$= \frac{e^{2t}}{2}\left(\begin{bmatrix} 2 \\ -1-i \end{bmatrix}(\cos t + i\sin t) + \begin{bmatrix} 2 \\ -1+i \end{bmatrix}(\cos t - i\sin t)\right) = e^{2t}\begin{bmatrix} 2\cos t \\ -\cos t + \sin t \end{bmatrix}$$

$$\frac{1}{2i}\left(\begin{bmatrix} 2 \\ -1-i \end{bmatrix} e^{(2+i)t} - \begin{bmatrix} 2 \\ -1+i \end{bmatrix} e^{(2-i)t}\right) = e^{2t}\begin{bmatrix} 2\sin t \\ -\cos t - \sin t \end{bmatrix}$$

General solution is a linear combination of these two,

$$\mathbf{x}(t) = a_1 e^{2t}\begin{bmatrix} 2\cos t \\ -\cos t + \sin t \end{bmatrix} + a_2 e^{2t}\begin{bmatrix} 2\sin t \\ -\cos t - \sin t \end{bmatrix}.$$

Then $\mathbf{x}(0) = \begin{bmatrix} 2a_1 \\ -a_1 - a_2 \end{bmatrix}$ and the initial conditions give $a_1 = 0.5$ and $a_2 = -1.5$. Finally

$$\underline{\mathbf{x}(t) = \begin{bmatrix} \cos t - 3\sin t \\ \cos t + 2\sin t \end{bmatrix}}$$

*Alternative solution* General solution has the form

$$\mathbf{x}(t) = c_1\mathbf{v}_1 \exp(\lambda_1 t) + c_2\mathbf{v}_2 \exp(\lambda_2 t),$$

where $\lambda_{1,2}$ are eigenvalues, $\mathbf{v}_{1,2}$ are corresponding eigenvectors and $c_{1,2}$ are some constants. The initial conditions imply

$$\begin{bmatrix}1\\1\end{bmatrix} = c_1 \begin{bmatrix}2\\-1-i\end{bmatrix} + c_2 \begin{bmatrix}2\\-1+i\end{bmatrix},$$

and solving the linear system, we obtain $c_1 = (1+3i)/4$, $c_2 = (1-3i)/4$. Then we have

$$\underline{\mathbf{x}(t)} = \frac{e^{2t}}{4}\left((1+3i)(\cos t + i\sin t)\begin{bmatrix}2\\-1-i\end{bmatrix} + (1-3i)(\cos t - i\sin t)\begin{bmatrix}2\\-1+i\end{bmatrix}\right)$$
$$= \frac{e^{2t}}{4}\Big((\cos t - 3\sin t + i(\sin t + 3\cos t))\begin{bmatrix}2\\-1-i\end{bmatrix}$$
$$+ (\cos t - 3\sin t - i(\sin t + 3\cos t))\begin{bmatrix}2\\-1+i\end{bmatrix}\Big) = \underline{e^{2t}\begin{bmatrix}\cos t - 3\sin t\\ \cos t + 2\sin t\end{bmatrix}}$$

**Problem 7** Suppose that $A$ is an $m \times n$-matrix with real entries. Prove that $\mathbf{x} \cdot A^T A\mathbf{x} \geq 0$ for each $\mathbf{x}$ in $\mathbb{R}^n$ and therefore each real eigenvalue of the matrix $A^T A$ is non-negative.

*Solution* First, note that $\mathbf{x} \cdot A^T A\mathbf{x} = \mathbf{x}^T A^T A\mathbf{x} = (A\mathbf{x})^T A\mathbf{x} = \|A\mathbf{x}\|^2 \geq 0$.

If $A^T A\mathbf{v} = \lambda\mathbf{v}$ for some real $\lambda$ and $\mathbf{v} \neq \mathbf{0}$ then $0 \leq \mathbf{v} \cdot A^T A\mathbf{v} = \mathbf{v} \cdot \lambda\mathbf{v} = \lambda\|\mathbf{v}\|^2$ and since $\|\mathbf{v}\|^2 > 0$ we conclude that $\lambda \geq 0$.