Norwegian University of Science and Technology
Department of Mathematical Sciences

Contacts during the exam:
Toke Meier Carlsen (46249940)
Louise Meier Carlsen (46366538)

Examination in TMA4115 Calculus 3

English
Wednesday June 5, 2013
Time: 09:00 – 13:00
Grades ready by June 26, 2013

Permitted aids (Code C): Specified, simple calculator (HP 30S or Citizen SR-270X)
Rottmann: *Matematisk formelsamling*

All answers must be justified, except in Problem 6, and your calculations should be detailed enough to clearly indicate your line of argument. Each of the problems 1a, 1b, 2a, 2b, 3a, 3b, 4, and 5 counts 10%, and each of the problems 6a–h counts 2.5%.

Problem 1 Let $T : \mathbb{R}^3 \to \mathbb{R}^3$ be a linear transformation such that

$$T\left(\begin{bmatrix}1\\0\\0\end{bmatrix}\right) = \begin{bmatrix}0\\4\\2\end{bmatrix},\ T\left(\begin{bmatrix}0\\1\\0\end{bmatrix}\right) = \begin{bmatrix}0\\2\\1\end{bmatrix},\ T\left(\begin{bmatrix}1\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\6\\3\end{bmatrix}.$$

a) Find the standard matrix A for the linear transformation T.

b) Find a basis for the null space, $\mathrm{Nul}(A)$, of A, and a basis for the column space, $\mathrm{Col}(A)$, of A.

Problem 2

a) Find the solution of the differential equation $y'' - y' = 0$ which satisfies $y(0) = 1$ and $y'(0) = -1$.

b) Find the general solution to the differential equation $y'' - y' = e^t \sin t$.

Problem 3 Let $\mathbf{u}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix}$.

a) Find an orthogonal basis for the plane in $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

b) Find the distance from $\mathbf{v}$ to the plane in $\mathbb{R}^3$ spanned by $\mathbf{u}_1$ and $\mathbf{u}_2$.

Problem 4 A particle moving in a plane under the influence of a force has the equation of motion

$$\mathbf{x}'(t) = \begin{bmatrix} 0 & -5 \\ 1 & -2 \end{bmatrix} \mathbf{x}(t),$$

where $\mathbf{x}(t)$ denotes the position of the particle at the time t. Find $\mathbf{x}(t)$ assuming that $\mathbf{x}(0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$. The answer should be given in the form $\mathbf{x}(t) = e^{at} \begin{bmatrix} c_1 \cos(bt) + c_2 \sin(bt) \\ c_3 \cos(bt) + c_4 \sin(bt) \end{bmatrix}$ where a, b, c_1, c_2, c_3 and c_4 are real numbers.

Problem 5 You are given that

$$\det\left(\begin{bmatrix} a & b & c \\ p & q & r \\ x & y & z \end{bmatrix}\right) = 2.$$

Use this information to compute the determinant of the matrix

$$\begin{bmatrix} x & y & z \\ p & q & r \\ 5p - 2a & 5q - 2b & 5r - 2c \end{bmatrix}.$$

Give reasons for your answer.

Problem 6 *You do not have to give reasons for your answers for this problem.*

a) For each of the following 4 complex numbers, determine whether it lies in the first quadrant of the complex plane (i.e., both its real part and its imaginary part are non-negative) or not.

1. $\sqrt{3} - i$.
2. $\frac{-2+i}{2+3i}$.
3. $e^{-2+7\pi i}$.
4. z^2, where $|z| = 2$ and $\text{Arg}(z) = \frac{\pi}{3}$.

b) Let A be an $n \times n$ matrix, B an $m \times n$ matrix, and C an $n \times m$ matrix, where $n \neq m$. For each of the following 4 expressions, determine whether it is well defined or not.

1. AB^T.
2. BB^T.
3. $CB + 2A$.
4. $B^2 - A^2$.

c) Let A and D be $n \times n$ matrices and let $\mathbf{b}$ be a nonzero vector in $\mathbb{R}^n$. For each of the following 4 statements, determine whether it is true or not.

1. If the system $A\mathbf{x} = \mathbf{b}$ has more than one solution, then the system $A\mathbf{x} = 0$ also has more than one solution.
2. If A^T is non-invertible, then A is non-invertible.
3. If $AD = I$, then $DA = I$.
4. If A has orthonormal columns, then A is invertible.

d) For each of the following 4 statements, determine whether it is true or not.

1. The two vectors $\begin{bmatrix} 3 \\ 2 \\ -5 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 0 \\ 1 \\ -1 \end{bmatrix}$ are orthogonal.
2. If $\mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$, $\mathbf{y} = \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{z} = \begin{bmatrix} -2 \\ 1 \\ -3 \end{bmatrix}$, then $\mathbf{z}$ belongs to the orthogonal complement of $\text{Span}\{\mathbf{x}, \mathbf{y}\}$.
3. An $m \times n$ matrix B has orthonormal columns if and only if $BB^T = I$.
4. If $\mathbf{x}$ is orthogonal to $\mathbf{y}$ and $\mathbf{z}$, then $\mathbf{x}$ is orthogonal to $\mathbf{y} - \mathbf{z}$.

e) Let A be an $n \times n$ matrix. For each of the following 4 statements, determine whether it is true or not.

1. If A is orthogonally diagonalizable, then A is symmetric.
2. If A is an orthogonal matrix, then A is symmetric.
3. If $\mathbf{x}^T A\mathbf{x} > 0$ for every $\mathbf{x} \neq \mathbf{0}$, then the quadratic form $\mathbf{x}^T A\mathbf{x}$ is positive definite.
4. Every quadratic form can by a change of variable be transformed into a quadratic form with no cross-product term.

f) Let $\mathbf{u}$ and $\mathbf{v}$ be nonzero vectors in $\mathbb{R}^n$, and let r be a scalar. For each of the following 4 statements, determine whether it is true or not.

1. $\|r\mathbf{v}\| = r\|\mathbf{v}\|$, unless $r = 0$.
2. If $\mathbf{u}$ and $\mathbf{v}$ are orthogonal, then $\{\mathbf{u}, \mathbf{v}\}$ is linearly independent.
3. If $\|\mathbf{u} + \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.
4. If $\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2$, then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal.

g) For each of the following 4 statements, determine whether it is true or not.

1. If A is a matrix, then $\text{rank}(A) = \dim(\text{Nul}(A))$.
2. A 5×10 matrix can have a 2-dimensional null space.
3. Row operations on a matrix can change its null space.
4. If the matrices A and B have the same reduced echelon form, then $\text{Row}(A) = \text{Row}(B)$.

h) For each of the following 4 statements, determine whether it is true or not.

1. The polynomials $p_1(t) = 1 + t^2$ and $p_2(t) = 1 - t^2$ are linearly independent.
2. If A is a 3×4 matrix, then the mapping $\mathbf{x} \mapsto A\mathbf{x}$ is a linear transformation from $\mathbb{R}^3$ to $\mathbb{R}^4$.
3. If a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ is onto $\mathbb{R}^4$ (or is surjective), then T cannot be one-to-one (injective).
4. A linear transformation $S : \mathbb{R}^5 \to \mathbb{R}^4$ cannot be one-to-one (injective).