**Problem 1 (Complex Numbers)**

**a)** Find all the solutions $z \in \mathbb{C}$ of the equation

$$z^3 = \mathrm{i}.$$

Sketch them (make a picture) in the complex plane. Write them in the form $z = a + \mathrm{i}b$ with $a$ and $b$ real, and in the form $z = r\mathrm{e}^{\mathrm{i}t}$ with $r$ and $t$ real.

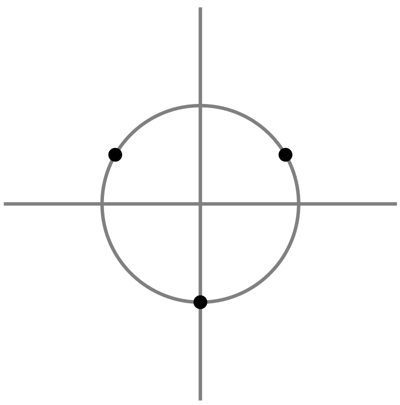
**b)** Write down Euler's formula for $\mathrm{e}^{\mathrm{i}t}$ with $t \in \mathbb{R}$.

**c)** An **addition formula** for a function $f$ is a formula that expresses $f(x+y)$ in terms of $f(x)$ and $f(y)$. Write down an addition formula for the complex exponential function $f(z) = \mathrm{e}^z$.

**d)** Let $z$ be a complex number, and let $\overline{z}$ denote its conjugate.
Show that $z + \overline{z}$ and $z \cdot \overline{z}$ are real numbers.

**Answer 1**

**a)** $z = -\mathrm{i} = \mathrm{e}^{\frac{3\pi}{2}\mathrm{i}}$ $\quad z = \frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2} = \mathrm{e}^{\frac{\pi}{6}\mathrm{i}}$ $\quad z = -\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2} = \mathrm{e}^{\frac{5\pi}{6}\mathrm{i}}$



**b)** $\mathrm{e}^{\mathrm{i}t} = \cos(t) + \mathrm{i}\sin(t)$

**c)** $f(x+y) = \mathrm{e}^{x+y} = \mathrm{e}^x\mathrm{e}^y = f(x) \cdot f(y)$

**d)** If $z = a + \mathrm{i}b$ with $a$ and $b$ real, then $z + \overline{z} = 2a$ and $z \cdot \overline{z} = a^2 + b^2$ are real.

**Problem 2** **(Systems of Linear Equations)**

**a)** Find all solutions to the equation

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 2 & 3 & 0 & 0 \\ 4 & 6 & 7 & 0 \\ 8 & 12 & 14 & 15 \end{bmatrix} \begin{bmatrix} t \\ x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 15 \\ 150 \end{bmatrix}.$$

**b)** For which values of $a, b, c, d, e, f, g \in \mathbb{R}$ do vectors of the form

$$v = \begin{bmatrix} a \\ b \\ c \\ d \\ e \\ f \\ g \end{bmatrix}$$

solve the equation

$$\begin{bmatrix} 0 & 1 & 2 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 4 \\ 0 & 0 & 0 & 0 & 0 & 1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} v = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}?$$

What is the dimension of the solution space?

**Answer 2**

**a)** The matrix is invertible, so that there is exactly one solution. Since the matrix is lower triangular, we can do backward substitution (first $t = 2$, then $2 \cdot 2 + 3x = 4$ implies $x = 0$, ...) to get

$$\begin{bmatrix} 2 \\ 0 \\ 1 \\ 8 \end{bmatrix}.$$

**b)** We have four free variables: $a$, $c$, $d$, and $g$, and the other components are given by the equations $b = -2c - 2g$, $e = -4g$, and $f = -3g$. The dimension is of the solution space is 4.

**Problem 3 (Spans, Linear Independence, Basis, Dimension)**

Consider the four vectors

$$v_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \\ 0 \end{bmatrix}, \; v_2 = \begin{bmatrix} 0 \\ 3 \\ 0 \\ 4 \end{bmatrix}, \; v_3 = \begin{bmatrix} 5 \\ 0 \\ 7 \\ 0 \end{bmatrix}, \; v_4 = \begin{bmatrix} 0 \\ 6 \\ 0 \\ 8 \end{bmatrix}.$$

**a)** Are the four vectors $v_1, v_2, v_3$, and $v_4$ linearly dependent or linearly independent? Justify your answer!

**b)** What is the dimension of the linear span of the four vectors $v_1, v_2, v_3$, and $v_4$? Justify your answer!

**c)** Write down a basis for the vector space spanned by the four vectors $v_1, v_2, v_3$, and $v_4$. Justify your answer!

**Answer 3**

**a)** The vectors are linearly dependent, because $v_4 = 2v_2$.

**b)** The vectors $v_1$, $v_2$, and $v_3$ are linearly independent since $v_1$ is not a multiple of $v_3$ and $v_2$ is not in the span of $\{v_1, v_3\}$. (There are many other ways to check this.) Therefore, their span is 3–dimensional.

**c)** Since $v_1$, $v_2$, and $v_3$ are linearly independent and span the vector space, they form a basis.

## Problem 4 (Matrix Algebra)

Suppose that $a, b, c$ and $x, y, z$ are arbitrary real numbers; you are not allowed to choose specific values!

**a)** Compute the matrix product

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}$$

in terms of $a, b, c$ and $x, y, z$.

**b)** What is the inverse of the matrix

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

in terms of $a, b, c$?

**c)** What is the determinant of the matrix

$$\begin{bmatrix} 1 & x & y \\ 0 & 1 & z \\ 0 & 0 & 1 \end{bmatrix}?$$

## Answer 4

**a)** Computing we get

$$\begin{bmatrix} 1 & a+x & b+y+az \\ 0 & 1 & c+z \\ 0 & 0 & 1 \end{bmatrix}.$$

**b)** For $x, y, z$ such $a + x = 0$, $b + y + az = 0$ and $c + z = 0$, the product obtained in **a)** is the identity, and thus, for this choice of $x, y, z$, the matrices are inverse to each other. We get

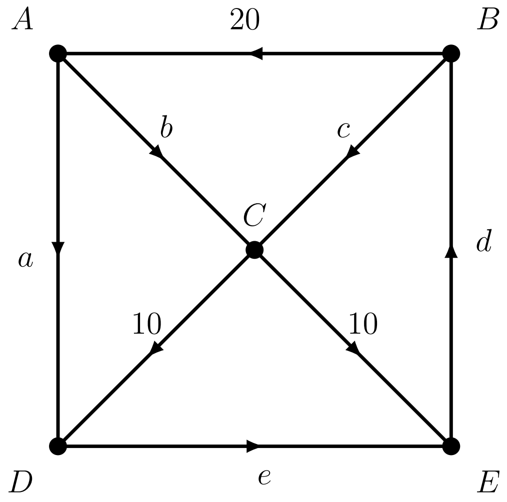
$$\begin{bmatrix} 1 & -a & ac-b \\ 0 & 1 & -c \\ 0 & 0 & 1 \end{bmatrix}.$$

A direct computation of the inverse if of course possible, but is more involved.

**c)** Since the matrix is an upper triangular matrix, the determinant is the product of the elements of the diagonal, and that is 1.

## Problem 5 (A Network and a Markov Process)

**a)** What is the system of linear equations that describes the flow in the following network?




**b)** When Sonja eats her evening meal, she has water, coffee, beer, or wine. She only has one beverage with her meal. She never has alcoholic beverages with two consecutive meals, but if she has beer or wine one time she is twice as likely to have coffee than water the next time. If she has water or coffee one evening, she has an alcoholic beverage the next time and is just as likely to choose beer as wine.

What is the stochastic matrix for this Markov chain?

## Answer 5

**a)**

$$\begin{array}{ll} A: & a+b=20 \\ B: & d-c=20 \\ C: & b+c=20 \\ D: & e-a=10 \\ E: & d-e=10 \end{array}$$

**b)**

$$\begin{bmatrix} 0 & 0 & 1/3 & 1/3 \\ 0 & 0 & 2/3 & 2/3 \\ 1/2 & 1/2 & 0 & 0 \\ 1/2 & 1/2 & 0 & 0 \end{bmatrix}$$

## Problem 6 (Eigenvalues and Eigenvectors)

**a)** Complete the following to a definition: "A number $\lambda$ is an **eigenvalue** of a square matrix $A$ if..."

**b)** Find one matrix $A$ such that

$$v_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

is an eigenvector of $A$ with eigenvalue 2, and

$$v_3 = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$$

is an eigenvector of $A$ with eigenvalue 3.

## Answer 6

**a)** "...there is a vector $v \neq 0$ such that $Av = \lambda v$." Also "$\det(A - \lambda I) = 0$" is correct.

**b)** There are many ways to arrive at

$$A = \begin{bmatrix} 0 & 1 \\ -6 & 5 \end{bmatrix}$$

For example, if we set

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then we just need to solve the system given by $Av_2 = 2v_2$ and $Av_3 = 3v_3$.

**Problem 7 (Second Order Linear Differential Equations)**

Suppose that $s$ is an arbitrary real constant. You are not allowed to choose a specific value.

**a)** Find the coefficients $p, q \in \mathbb{R}$ such that the second order linear differential equation

$$y''(t) + py'(t) + qy(t) = 0$$

has the solution $y_s$ with

$$y_s(t) = \cos(s + t).$$

**b)** What are the initial conditions for the solution $y_s$ at time $t = 0$ (in terms of $s$)?

**c)** Write $y_s(t)$ as a linear combination of $\cos(t)$ and $\sin(t)$, (with coefficients depending on $s$).

**Answer 7**

**a)** First we compute $y_s'(t) = -\sin(s + t)$ and $y_s''(t) = -\cos(s + t)$. Then we can insert these into the differential equation to find that $p = 0$ and $q = 1$, and thus that the equation becomes $y'' + y = 0$.

**b)** We find $y_s(0) = \cos(s)$ and $y_s'(0) = -\sin(s)$.

**c)** The solution is $\cos(s + t) = \cos(s)\cos(t) - \sin(s)\sin(t)$. There are several ways to arrive here. For example, we can use trigonometric formulas. Or, we can the general solution of the differential equation given by

$$y(t) = A\sin(t) + B\cos(t)$$

and use the initial conditions for $y_s(t)$ found in **b)** to get that $A = -\sin(s)$ and $B = \cos(s)$.

**Problem 8 (Systems of Linear Differential Equations)**

Consider the linear system

$$\begin{aligned} x'(t) &= x(t) + y(t) \\ y'(t) &= y(t) + x(t) \end{aligned}$$

of first order differential equations. Given $x_0, y_0 \in \mathbb{R}$, consider the initial conditions

$$\begin{aligned} x(0) &= x_0 \\ y(0) &= y_0. \end{aligned}$$

**a)** Find the solution of the given system with the given initial conditions.

**b)** Check that the solution found satisfies all four equations.

**Answer 8**

**a)** There are several ways of arriving at

$$\begin{aligned} x(t) &= \frac{x_0}{2}(\mathrm{e}^{2t} + 1) + \frac{y_0}{2}(\mathrm{e}^{2t} - 1) = \frac{x_0 + y_0}{2}\,\mathrm{e}^{2t} + \frac{x_0 - y_0}{2}, \\ y(t) &= \frac{x_0}{2}(\mathrm{e}^{2t} - 1) + \frac{y_0}{2}(\mathrm{e}^{2t} + 1) = \frac{x_0 + y_0}{2}\,\mathrm{e}^{2t} + \frac{y_0 - x_0}{2}. \end{aligned}$$

For example, the matrix of the system is given by

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

which has eigenvalues $\lambda_1 = 0$ and $\lambda_2 = 2$ with corresponding eigenvectors $v_1$ and $v_2$, say. The solution is given by

$$e^{At} \begin{bmatrix} x_0 \\ y_0 \end{bmatrix}.$$

Another option is to use the formula for the general solution given by

$$C_1 v_1 e^{\lambda_1 t} + C_2 v_2 e^{\lambda_2 t}$$

and find $C_1$ and $C_2$ using the initial conditions.

**b)** This is a direct computation from the solution.

**Problem 9 (The Least-Squares Method)**

Consider the four data points

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ t \end{bmatrix}.$$

in 3–space, where $t$ is a real parameter.

**a)** Write down the equations

$$A \begin{bmatrix} p \\ q \\ r \end{bmatrix} = b$$

that would be satisfied if the function $z = px + qy + r$ went through the four data points. State explicitly what $A$ and $b$ are!

**b)** Can you choose $t$ so that the equations in **a)** have a solution? Justify your answer!

**c)** Suppose that the parameter $t$ is given so that the equations in **a)** do not have a solution. Explain what a **least-squares "solution"** is. (You are not asked to find it for real!)

**Answer 9**

**a)** We substitute in $z = px + qy + r$ the values of $x, y, z$ given by the vectors to get

$$\begin{aligned} 1 &= r \\ 2 &= p + r \\ 3 &= q + r \\ t &= p + q + r \end{aligned} \qquad A = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \qquad b = \begin{bmatrix} 1 \\ 2 \\ 3 \\ t \end{bmatrix}.$$

**b)** The first three equations imply $r = 1$, $p = 1$, and $q = 2$. Then the fourth equation is satisfied if and only if $t = 4$, so that for $t \neq 4$ there is no solution.

**c)** A least-squares "solution" to $Ax = b$ is a solution to $A^\top Ax = A^\top b$. This finds the $x$ such that $\|Ax - b\|^2$ is minimal.

## Problem 10 (Euclidean Geometry)

**a)** Let

$$u = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix},$$

so that $u_j = 1$ for all $j = 1, \ldots, n$.
What is the Euclidean length, the norm $\|u\|$, of the vector $u$?

**b)** Let $u$ be as in **a)** and let

$$v = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

Find the co-ordinates of the orthogonal projection of the vector $v$ onto the line spanned by $u$.

**c)** Let $w \neq 0$ be a non-zero vector in $\mathbb{R}^n$, and let $H = \text{Span}\{w\}^\perp$ be the set of vectors that are orthogonal to $w$. Give a formula for the orthogonal projection $\mathbb{R}^n \to \mathbb{R}^n$ onto $H$.

## Answer 10

**a)**

$$\|u\| = \sqrt{n}$$

**b)** All co-ordinates are equal to $\frac{1}{n}(v_1 + \cdots + v_n)$, because

$$\text{pr}_u(v) = \frac{\langle v, u \rangle}{\langle u, u \rangle} u = \frac{v_1 + \cdots + v_n}{n} u.$$

**c)**

$$\text{pr}_H(v) = v - \frac{\langle v, w \rangle}{\langle w, w \rangle} w$$