A Theory of the Amplitude of Free and Forced Triode Vibrations.

By BALTH. van der POL, Jun., D.Sc.

INTRODUCTION.

The conditions to be fulfilled by a circuit linked to a triode in order to generate oscillations have been fully treated by several writers. Their differential equations were mostly confined however to the linear terms and in consequence could be satisfied by any amplitude whatever, though implicitly these differential equations are only valid for infinitesimal oscillations.

As a given triode oscillator, with definite settings of the circuit constants, will generate oscillatory currents with harmonics all having a definite amplitude, it may be worth while to put forward a theory of the oscillating triode having regard to the non-linear terms in the equations. Only in this way can certain properties of a triode system be explained which would otherwise escape the analytical treatment, such as the rectifying action, the interaction of two oscillatory systems, the function of a thermionic bulb as a limiter of amplitude, the working of the heterodyne, the greater magnifications given to small impressed E.M.F.'s than to bigger ones, the generation of higher harmonics, etc.

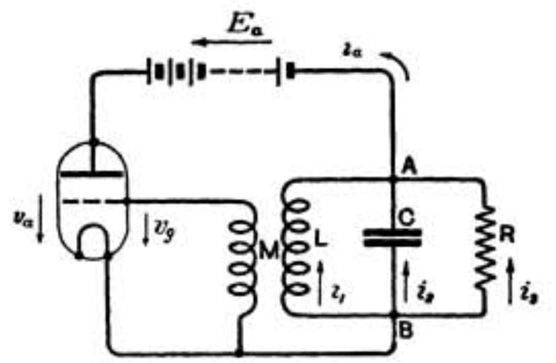

FIG. 1.

When the non-linear terms are retained in the equations the latter, and still more their solutions, soon become very complicated and in order to show clearly and definitely the importance of these terms it seems advisable to treat analytically that system of connections which renders the equations as simple as possible, thus obviating as far as possible, purely analytical complications, and allowing the physical meaning of the formulæ to be clearly seen. This is especially the case in locating the resistance of the oscillatory L C flywheel circuit connected to the anode and filament, not in series either with the self-inductance or capacitance but in parallel to both.

I.—THE TRIODE AS A GENERATOR OF OSCILLATIONS.

1. Let, as indicated in Fig. 1

i_1 be the total current in the self-inductance branch L.

i_2 be the total current in the capacity branch C.

i_3 be the total current in the resistance branch R.

$i_a = i_1 + i_2 + i_3$ be the total anode current.

E_a the E.M.F. of the anode battery.

v_a the plate potential difference, v_g the grid potential difference } with respect to the filament

then we have

$$L\frac{di_1}{dt} = Ri_3 = \frac{1}{C}\int i_2 dt = E_a - v_a$$

$$M\frac{di_1}{dt} = v_g$$

where M is chosen positive when it has the proper sense to generate oscillations. Eliminating i_1, i_2 and i_3 we arrive at the simple equation:

$$\frac{di_a}{dt} + \left(\frac{1}{L} + \frac{1}{R}\frac{d}{dt} + C\frac{d^2}{dt^2}\right)(v_a - E_a) = 0 \quad . \quad . \quad . \quad . \quad . \quad (1)$$

where the grid currents are neglected.

Let further the anode current i_a be a function ϕ of the *single* variable $v_a + gv_g$ (called the "lumped voltage" *) where g is the "voltage ratio" of the tube. Hence

$$i_a = \phi(v_a + gv_g) = \phi\left\{v_a + g\frac{M}{L}(E_a - v_a)\right\} \quad . \quad . \quad . \quad (2)$$

In the steady, though unstable, state $v_{a_0} = E_a$ as there is no potential drop in the resistanceless branch L. The steady (unstable) anode current we call i_{a_0}; obviously

$$i_{a_0} = \phi(v_{a_0}) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (3)$$

If v be the momentary deviations of the plate potential from the unstable value E_a we have $v_a - v_{a_0} = v$, and the total anode current becomes:

$$i_a = \phi\left\{v_{a_0} - \left(g\frac{M}{L} - 1\right)v\right\}$$

or

$$i_a = \phi(v_{a_0} - kv) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (4)$$

where $$k = \left(g\frac{M}{L} - 1\right)$$

* W. H. Eccles, RADIO REVIEW, 1, p. 73, November, 1919.

If i be the instantaneous deviations of the total plate current i_a from the unstable value i_{a_0}, we have

$$i = i_a - i_{a_0} = \phi\,(v_{a_0} - kv) - \phi\,(v_{a_0}) \quad . \quad . \quad . \quad . \quad . \quad (5)$$

which may be written

$$i = \psi\,(kv) \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (6)$$

The fundamental equations of the triode oscillator are therefore

$$\left.\begin{aligned} &\frac{di}{dt} + C\frac{d^2v}{dt^2} + \frac{1}{R}\frac{dv}{dt} + \frac{1}{L}\,v = 0 \\ &i = \psi\,(kv) \end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad (\mathrm{I.}^{a,\,b})$$

2. Before proceeding we will first consider the functions ϕ and ψ.

The steady characteristic $i_a = \phi\,(E_a)$ has the well-known form of Fig. 2. From this characteristic the one of Fig. 3 can readily be derived where i is represented as $i = \psi\,(kv) = \phi\,(v_{a_0} - kv) - \phi\,(v_{a_0})$.

In the latter figure this function has been drawn for the values $k = -1, \frac{1}{2}, 1$ while for $k = 0$, $\psi\,(kv)$ coincides with the axis of abscissæ.

From this figure, which may be said to represent a family of "derived" characteristics, it is clear that the "falling" characteristic necessary for the generation of oscillations is only obtained when $k > 0$, *i.e.*, for

$$g\,\frac{M}{L} > 1.$$

In the second part of this paper it will be shown more fully that, if $k = 0$, *i.e.*, $g\,\dfrac{M}{L} = 1$, the "resistance" brought into the circuit by the presence of the triode is just compensated for by its regenerative action.

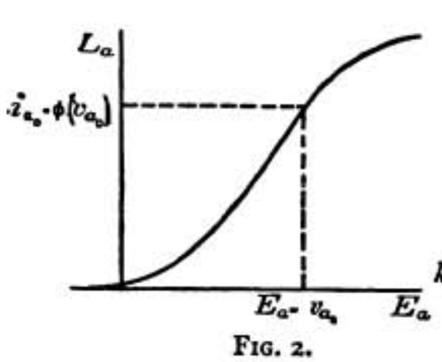

Fig. 2.

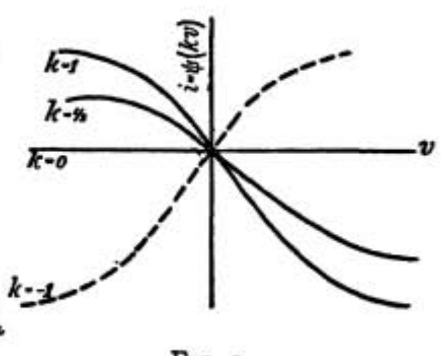

Fig. 3.

3. Taking up the analysis again, we see from (2), (4) and (5) that if we assume that the curves in Fig. 3 can be represented by the equation

$$i = \psi\,(kv) = -\alpha v + \beta v^2 + \gamma v^3 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (7)$$

then by Maclaurin's theorem,

$$\left.\begin{aligned}\alpha &= k\left(\frac{\partial i_a}{\partial v_a}\right)_{v_a = E_a}\\ \beta &= \frac{k^2}{2}\left(\frac{\partial^2 i_a}{\partial v_a{}^2}\right)_{v_a = E_a}\\ \gamma &= -\frac{k^3}{6}\left(\frac{\partial^3 i_a}{\partial v_a{}^3}\right)_{v_a = E_a.}\end{aligned}\right\} \quad \ldots\ldots\ldots \quad (8)$$

where α and γ may be taken positive as long as $k > 0$. If $E_a = v_{a_0}$ is chosen such that i_{a_0} is just half the saturation value while the characteristic is considered symmetrical with regard to the point (v_{a_0}, i_{a_0}), β will vanish and (I.[a, b]) will be reduced to

$$C\frac{d^2v}{dt^2} + \left(\frac{1}{R} - \alpha\right)\frac{dv}{dt} + \frac{1}{L}v + \beta\frac{d(v^2)}{dt} + \gamma\frac{d(v^3)}{dt} = 0 \quad \ldots \quad (\text{II.})$$

If our considerations are limited to small oscillations only, both γ and β may be neglected, thus leaving a linear equation with the condition for the generation of an alternating current, that

$$\frac{1}{R} - \alpha = 0.$$

With

$$\left(\frac{\partial i_a}{\partial v_a}\right)_{v_a = E_a} = h_a = \frac{\alpha}{k},$$

$$\left(\frac{\partial i_a}{\partial v_g}\right)_{v_g = 0} = h_g,$$

and

$$g = \frac{h_g}{h_a},$$

this amounts to

$$g\frac{M}{L} - 1 = \frac{1}{Rh_a} \quad \ldots\ldots\ldots \quad (9)$$

However, with M greater than the value given by (9) the coefficient of $\frac{dv}{dt}$ in II. becomes negative and the oscillations would build up indefinitely were it not for the presence of the non-linear terms (with γ and β) in II. which put a limit to this increase, or, speaking geometrically: it is the curvature of the characteristic which determines the final amplitude.

4. First Method of finding the Amplitude of the Fundamental.

Our equation II. is closely related to some problems which arise in the analytical treatment of the perturbations of planets by other planets, and of the vibrations of bodies not obeying Hooke's law, upon which Helmholtz's well-known theory of combination tones was based. Hence a somewhat similar way of solving II. may be applied here.*

* In this paragraph we follow closely a method of solution given by Prof. Lorentz in a series of lectures at Leiden University.

Rewriting II. as:

$$\left(\frac{d^2}{dt^2}+\frac{1}{CL}-\epsilon\right)v+\left\{\epsilon+\left(\frac{1}{CR}-\frac{\alpha}{C}\right)\frac{d}{dt}\right\}v+\frac{\beta}{C}\frac{d(v^2)}{dt}+\frac{\gamma}{C}\frac{d(v^3)}{dt}=0\ . \quad (10)$$

we introduce a first order correction ϵ to the square of the natural angular frequency ω^2 by putting

$$\omega^2=\frac{1}{CL}-\epsilon\ . \quad (11)$$

instead of

$$\omega^2=\frac{1}{CL}$$

Since the voltage is the sum of the fundamental and the various harmonics, we put

$$v=v_1+v_2+v_3+\dots \quad (12)$$

and assume that this series converges, v_1 having the fundamental frequency, v_2 being the first overtone, v_3 the second, etc. Moreover v_1 will be taken to be of the first order of magnitude, v_2 of the second, v_3 of the third, etc.

Putting (12) in (10) we obtain an equation with several terms of different orders of magnitude.

Now equating to zero the sum of all terms of the first order, the sum of all terms of the second order, also of the third order, and limiting ourselves to terms of the third order, it would be impossible to get a solution unless we also assumed $\left\{\left(\frac{1}{CR}-\frac{\alpha}{C}\right)\frac{d}{dt}+\epsilon\right\}$ to be of the second order. With this assumption the following equations are obtained:

$$\left(\frac{d^2}{dt^2}+\frac{1}{CL}-\epsilon\right)v_1=0 \quad (13)$$

$$\left(\frac{d^2}{dt^2}+\frac{1}{CL}-\epsilon\right)v_2=-\frac{\beta}{C}\frac{d}{dt}(v_1^2) \quad (14)$$

$$\left(\frac{d^2}{dt^2}+\frac{1}{CL}-\epsilon\right)v_3=-\left\{\epsilon+\left(\frac{1}{CR}-\frac{\alpha}{C}\right)\frac{d}{dt}\right\}v_1-\frac{2\beta}{C}\frac{d(v_1v_2)}{dt}-$$
$$-\frac{\gamma}{C}\frac{d(v_1^3)}{dt} \quad (15)$$

From (13) which is an equation of free oscillations we have

$$v_1=a\cos\omega t \quad \text{with } \omega^2=\frac{1}{CL}-\epsilon \quad (16)$$

where a is the unknown amplitude and ω the unknown angular frequency. Equation (14), representing a forced vibration of period 2ω, yields with (16)

$$v_2=-\frac{a^2\beta}{3C\omega}\sin 2\omega t\ . \quad (17)$$

Equation (15) represents a forced vibration of periods ω and 3ω. In order to make sure that v_3 shall only contain terms of the frequency 3ω that part of the impressed E.M.F. in (15) having the frequency ω may now be put equal to zero, thus yielding two equations for the frequency correction ϵ and the fundamental amplitude a.*

We thus find from (15):

$$\left\{\epsilon+\left(\frac{1}{CR}-\frac{\alpha}{C}\right)\frac{d}{dt}\right\} a\cos\omega t+\frac{2\beta}{C}\frac{d}{dt}\left(-\frac{a^3\beta}{6C\omega}\sin\omega t\right)+\frac{\gamma}{C}\frac{d}{dt}\left(\frac{3}{4}a^3\cos\omega t\right) \equiv 0 \quad . \quad . \quad . \quad . \quad . \quad (18)$$

which must be satisfied at any moment. Hence we obtain for the frequency correction

$$\epsilon=\frac{a^2\beta^2}{3C^2} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (19)$$

and for the square of the amplitude of the fundamental

$$\boldsymbol{a^2=\frac{4}{3\gamma}\left(\alpha-\frac{1}{R}\right)} \quad . \quad . \quad . \quad . \quad . \quad (20)$$

Finally, after solving also v_3, the solution of II. is:

$$v=a\cos\omega t-\frac{a^2\beta}{3C\omega}\sin 2\omega t+a^3\left(\frac{\beta^2}{8\omega^2C^2}\cos 3\omega t+\frac{3\gamma}{32\omega C}\sin 3\omega t\right) \quad . \quad . \quad (21)$$

while i contains the steady component $\frac{1}{2}a^2\beta$ which is the shift of the indication of a direct current milliamperemeter in the anode circuit, observed when the system starts generating oscillations.

A steady component does not occur in v which is obvious from the fact that one branch (L) has no resistance.

5. Second Method of finding the Amplitude of the Fundamental.

Instead of finding a solution v which at any moment satisfies (I.^{a,b}) we may from (I.^{a,b}) first derive several energy equations.

Integrating one such equation over the fundamental period gives an equation out of which the time t has disappeared. Next we try to find a quadratic mean value of v satisfying this mean energy equation and thus obtain the value of the amplitude a.

Very suitable for this way of treating the problem is the equation obtained by multiplying (I.^{a}) by $\int v dt$ and then integrating the result over the unknown fundamental period T.

Integrating by parts yields us the simple energy-equation (the other terms vanishing)

$$\int_0^T iv dt+\frac{1}{R}\int_0^T v^2 dt=0 \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (22)$$

* If those terms in the second member of (15) which contain the frequency ω were not identified with zero, so called "secular terms" in v_3 would be necessary (of the form $t\sin\omega t$) which would disturb the purely periodical character of our solution.

We now assume as a first approximation that v varies sinusoidally, hence

$$v = a \cos \omega t$$

$$\therefore \quad \frac{1}{T}\int_0^T v^2 dt = \frac{a^2}{2}$$

Further

$$iv = -\alpha v^2 + \beta v^3 + \gamma v^4$$

yielding

$$\frac{1}{T}\int_0^T iv dt = -\frac{\alpha}{2}a^2 + \frac{3}{8}\gamma a^4.$$

Putting these results in (22) we at once obtain the square of the amplitude:

$$a^2 = \frac{4}{3}\frac{\alpha - \frac{1}{R}}{\gamma}$$

which equals (20) found above.

The assumption (in order to find the amplitude) of v having an accurately sinusoidal form may be justified by the following considerations.

FIG. 4. FIG. 5. FIG. 6.

If v is assumed to have the form of Fig. 5 with maximum elongation a', the mean quadratic value of v is found from (22) to be

$$a'^2 = \frac{5}{3}\frac{\alpha - \frac{1}{R}}{\gamma}$$

while the fundamental harmonic component a is related to a' by the equation

$$a = \frac{8}{\pi^2}a'.$$

Hence for the waveform of Fig. 5 we find

$$a^2 = \frac{64}{\pi^4}\cdot\frac{5}{3}\frac{\alpha - \frac{1}{R}}{\gamma} = 1{\cdot}095\frac{\alpha - \frac{1}{R}}{\gamma}$$

or

$$a = 1{\cdot}047\frac{\alpha - \frac{1}{R}}{\gamma}.$$

For the widely different waveform of Fig. 6 with maximum elongation a'', we find in a similar way

$$a''^2 = \frac{\alpha - \frac{1}{R}}{\gamma}$$

$$a = \frac{4}{\pi} a''$$

$$\therefore a^2 = \frac{16}{\pi^2} \frac{\alpha - \frac{1}{R}}{\gamma} = 1{\cdot}620 \frac{\alpha - \frac{1}{R}}{\gamma}$$

or

$$a = 1{\cdot}274 \frac{\alpha - \frac{1}{R}}{\gamma}.$$

Taking the mean of the two a's from Fig. 5 and Fig. 6 we find a value that differs less than a half of one per cent. from the value of a obtained from the sinusoidal wave form, namely

$$a = 1{\cdot}155 \frac{\alpha - \frac{1}{R}}{\gamma}.$$

Hence we see that the extreme forms of Fig. 5 and Fig. 6 yield amplitudes for the fundamental frequency of v of values only slightly different from the one obtained on the basis of the purely sinusoidal v, and we may accept the result

$$a^2 = \frac{4}{3} \frac{\alpha - \frac{1}{R}}{\gamma} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (20)$$

with confidence.

6. Discussion of results so far obtained.—Formula (20) only leads to a stable oscillation with real amplitude when, for $\gamma > 0$, $\alpha > \frac{1}{R}$, *i.e.*, for

$$\left(g\frac{M}{L} - 1\right) > \frac{1}{Rh_a}$$

a^2 would also be positive for both $\gamma < 0$ and $\alpha - \frac{1}{R} < 0$. However the oscillations set up in this case are unstable * so that this case can further be excluded from our physical considerations.

The damping factor of the L C R circuit of Fig. 1 (see also Fig. 7) but disconnected from the triode, is well known to be $\frac{1}{2CR}$. Hence the smaller the shunt resistance the more the natural oscillations of the circuit L C R

* Rayleigh, *Philosophical Magazine*, April, 1883.

will be damped. Similarly the damping factor of a circuit consisting of L, C and r *in series*, is $\frac{r}{2L}$.

If therefore the latter circuit were connected to the triode, we can, with the accuracy obtained in the above analysis, simply replace $\frac{1}{R}$ by $\frac{Cr}{L}$ and hence obtain for the amplitude of oscillations of such a series resistance circuit

$$a^2 = \frac{4}{3}\frac{\alpha - \frac{Cr}{L}}{\gamma}.$$

The same formula will be found if, *ab initio*, we start formulating the differential equations for the case of a series resistance r. The analysis however is much more complicated.

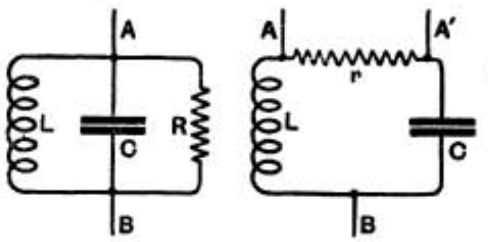

FIG. 7.

As a first approximation it does not matter whether the plate is connected at A or A′ (Fig. 7), whether therefore the resistance r is in the L or the C branch. For the sake of simplicity we shall discuss our further results, with the *parallel* resistance R, keeping in mind that the *smaller* is R, the *more* the system is damped.

From (19) and (20) it follows that the frequency correction ϵ, as might be expected, is a function of the amplitude, but also of β, *i.e.*, of $\frac{\partial^2 i_a}{\partial v_a^2}$, whereas the amplitude, as a first approximation, is not affected by the magnitude of β, but only depends upon α and γ, *i.e.*, upon $\frac{\partial i_a}{\partial v_a}$ and $\frac{\partial^3 i_a}{\partial v_a^3}$.

This dependence of the frequency upon the part of the characteristic where the triode is functioning may be illustrated by the following experiment.

It is well known that only with the utmost care can such a small correction experimentally be made evident by altering *e.g.* the grid potential by means of a potentiometer arrangement, as the slightest alteration of the circuits materially affects the tuning conditions. Hence it appears a better course to let the triode itself bring its oscillations on a different part of its characteristic. To this end it is only necessary, instead of connecting the grid directly to the grid coupling coil of M, to insert at this place a large condenser of say 1 μF.

If the insulating quality of this condenser is not extreme, oscillations will be set up which gradually die down owing to the negative charge accumulating on the grid, the latter thus acquiring a gradually increasing negative potential. After some time (occasionally twenty seconds) the system will suddenly again burst into oscillations and the same phenomena will occur over and over again. If in another (autodyne) system one listens to this phenomena, the audible combination tone is very distinctly heard to alter its pitch during the time the first system oscillates, thus proving clearly the gradual fall of frequency when a triode system vibrates lower down on its characteristic.

The Capacity of Rectangular Plates and a Suggested Formula for the Capacity of Aerials.

By THE EDITOR.

The capacity of any conducting body, whatever its shape, can be determined by the method devised by the author for the calculation of the capacity of multiple wire antennæ. A uniform distribution of charge over the surface is assumed and the average potential calculated; the uniform potential actually obtained when the same total charge is allowed to have its natural distribution will approximate very closely to this average potential. Formulæ can be established for the value of the average potential over surfaces of

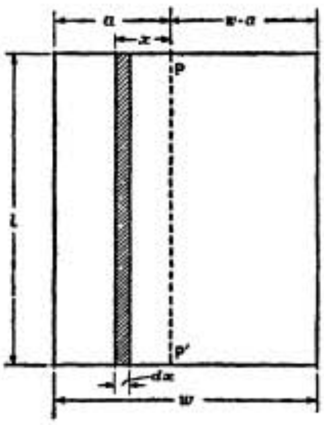

FIG. 1.

simple geometrical form which greatly simplify the calculation. The author has also shown that the same method and formulæ can be used to calculate the resistance to earth of any system of buried conductors. In working out the original formulæ it was assumed that the conductors consisted of round wires, which is of course usually the case with aerials, but in the case of buried conductors copper strip is often employed and the formulæ for round wire cannot be applied. It is, moreover, often necessary to determine approximately the capacity of rectangular plates.