**Problem 1** Let $p(z) = z^3 + 27$. Find all the roots of $p(z)$, write the roots in standard form and sketch the roots in the complex plane.

**Problem 2** Consider the points $(1,5), (-1,9)$ og $(2,12)$ in $\mathbb{R}^2$.

**a)** Find a second order polynomial $p(x) = ax^2 + bx + c$ that passes through these points.

**b)** Use the method of least squares to find the first order polynomial $q(x) = dx + e$ which best fits these points.

**Problem 3** Let

$$A = \begin{bmatrix} 1 & 0 & 4 \\ -2 & 1 & -5 \\ 2 & -2 & a^2 - 2 \end{bmatrix}, \qquad \text{where } a \in \mathbb{R}.$$

**a)** For which real numbers $a$ is the matrix $A$ invertible?

**b)** For which real numbers $a$ does the equation

$$A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ a-4 \end{bmatrix}$$

have

- no solutions?
- exactly one solution?
- infinitely many solutions?

**Problem 4** Let $\mathcal{M}_2(\mathbb{R})$ be the vector space of real $2 \times 2$-matrices.
Consider the following subset of $\mathcal{M}_2(\mathbb{R})$:

$$U = \{A \in \mathcal{M}_2(\mathbb{R}) \mid A^T = -A\}.$$

In other words, $U$ consists of all real $2 \times 2$-matrices $A$ such that $A^T = -A$.

**a)** Show that $U$ is a subspace of $\mathcal{M}_2(\mathbb{R})$.
*Hint: It might be useful to know that* $(A + B)^T = A^T + B^T$ *and* $(cA)^T = c \cdot A^T$.

**b)** Find a basis for $U$ and determine the dimension of $U$.

**Problem 5** A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ is given by

$$T\left(\begin{bmatrix}0\\2\\1\end{bmatrix}\right) = \begin{bmatrix}2\\1\\3\end{bmatrix}, \quad T\left(\begin{bmatrix}0\\2\\2\end{bmatrix}\right) = \begin{bmatrix}4\\4\\2\end{bmatrix}, \quad T\left(\begin{bmatrix}2\\1\\1\end{bmatrix}\right) = \begin{bmatrix}2\\0\\2\end{bmatrix}.$$

Find the standard matrix $A$ of the linear transformation $T$, and determine if $T$ is injective and/or surjective.

**Problem 6**

**a)** Show that

$$\langle \mathbf{x}, \mathbf{y} \rangle = \langle \begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}, \begin{bmatrix}y_1\\y_2\\y_3\end{bmatrix} \rangle = x_1y_1 + 4x_2y_2 + 9x_3y_3$$

defines an inner product on $\mathbb{R}^3$.

**b)** Find an orthogonal basis for the subspace of $\mathbb{R}^3$ spanned by the vectors

$$\left\{ \begin{bmatrix}2\\-1\\1\end{bmatrix}, \begin{bmatrix}-1\\2\\-1\end{bmatrix} \right\}$$

with respect to the inner product in a).

**Problem 7**

Let

$$A = \begin{bmatrix}a & b-a\\0 & b\end{bmatrix} \text{ where } a, b \in \mathbb{R} \text{ and } a \neq b.$$

**a)** Find an invertible matrix $P$ and a diagonal matrix $D$ such that $A = PDP^{-1}$ and determine an expression for $A^k$, where $k$ is an arbitrary positive integer.

**b)** Find the general solution of the following system of differential equations

$$\mathbf{y}' = \begin{bmatrix}-2 & 5\\0 & 3\end{bmatrix}\mathbf{y}$$

and then find the solution of the system which satisfies $y_1(0) = 2$ and $y_2(0) = -4$.

**Problem 8** Find the general solution of $y'' + 5y' + 6y = e^{-2t}$ and solve for the initial conditions $y(0) = y'(0) = 1$.

**Problem 9** Let $\mathcal{P}_2(\mathbb{R})$ be the vector space of polynomials of degree less than or equal to 2, and with real coefficients.

Let $S = \{p_1(x), p_2(x), p_3(x)\}$ with

$$p_1(x) = x^2 + 1, \qquad p_2(x) = 6x^2 + x + 2, \qquad p_3(x) = 3x^2 + x.$$

The set $S$ forms a basis for $\mathcal{P}_2(\mathbb{R})$ (you do not have to prove this).
Find the coordinate vector of $q(x) = x^2 + 2x + 3$ with respect to the basis $S$.

**Problem 10** Let $V$ be an inner product space and let $\mathbf{v}_1$ og $\mathbf{v}_2$ be two vectors in $V$ such that both $\mathbf{v}_1 \neq \mathbf{0}$ and $\mathbf{v}_2 \neq \mathbf{0}$. Show that if $\mathbf{v}_1$ and $\mathbf{v}_2$ are orthogonal, then they are also linearly independent.