Eksamensoppgave B–29, Fresnelintegraler

Oppgavetekst

Gitt $f(z) = \mathrm{e}^{iz^2}$, $R > 0$ og

$$\Gamma_R : \theta \mapsto R\mathrm{e}^{i\theta}, \quad 0 \le \theta \le \tfrac{\pi}{4}.$$

- Vis at
$$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,\mathrm{d}z = 0.$$
Hint: $\sin 2\theta \ge 4\theta/\pi, 0 \le \theta \le \pi/4$.
- La $C_R = [0, R] \cup \Gamma_R \cup K$ der
$$K : t \mapsto (R - t)\,\mathrm{e}^{i\pi/4}, \quad 0 \le t \le R$$
Finn $I_c = \int_0^\infty \cos(t^2)\,\mathrm{d}t$ og $I_s = \int_0^\infty \sin(t^2)\,\mathrm{d}t$ ved å integrere én gang rundt C_R og la $R \to \infty$. Hint: $\int_0^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t = \sqrt{\pi}/2$.

Theorem (*ML*-ulikheten)

$$\left|\int_C f(z)\,\mathrm{d}z\right| \leq ML$$

der

$$M = \max\{|f(z)| : z \in C\}, \quad L = \textit{lengden av } C.$$

- I dette tilfellet er $L = \pi R/4$.
- Estimat av $|f(z)|$ langs Γ_R:

$$f(z) = \mathrm{e}^{i(R\mathrm{e}^{i\theta})^2} = \mathrm{e}^{iR^2(\cos 2\theta + i\sin 2\theta)} = \mathrm{e}^{-R^2\sin 2\theta}\,\mathrm{e}^{iR^2\cos 2\theta}$$

$$|f(z)| = \mathrm{e}^{-R^2\sin 2\theta} \leq \mathrm{e}^{-R^2\,4\theta/\pi} \leq 1, \quad 0 \leq \theta \leq \pi/4$$

- Fra *ML*-ulikheten får vi da $\left|\int_{\Gamma_R} f(z)\,\mathrm{d}z\right| \leq \pi R/4$. Ubegrenset for $R \to \infty$. *ML*-ulikheten for grov i dette tilfellet.

$\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,\mathrm{d}z = 0$, mer presist

Vi parametriserer linjeintegralet

$$\int_{\Gamma_R} f(z)\,\mathrm{d}z = \int_0^{\pi/4} f(R\mathrm{e}^{i\theta})\, iR\mathrm{e}^{i\theta}\,\mathrm{d}\theta = iR\int_0^{\pi/4} f(R\mathrm{e}^{i\theta})\,\mathrm{e}^{i\theta}\,\mathrm{d}\theta,$$

og får da

$$\Big|\int_{\Gamma_R} f(z)\,\mathrm{d}z\Big| \le R\int_0^{\pi/4} |f(R\mathrm{e}^{i\theta})|\,\mathrm{d}\theta \le R\int_0^{\pi/4} \mathrm{e}^{-R^2 4\theta/\pi}\,\mathrm{d}\theta$$

$$= R\Big[-\frac{\pi}{4R^2}\,\mathrm{e}^{-R^2 4\theta/\pi}\Big]_0^{\pi/4} = \frac{\pi}{4R}\,(1-\mathrm{e}^{-R^2}).$$

Det betyr at

$$0 \le \Big|\int_{\Gamma_R} f(z)\,\mathrm{d}z\Big| \le \frac{\pi}{4R}(1-\mathrm{e}^{-R^2})$$

og ved "klemmeloven" er da $\lim_{R\to\infty} |\int_{\Gamma_R} f(z)\,\mathrm{d}z| = 0$ som betyr at $\lim_{R\to\infty} \int_{\Gamma_R} f(z)\,\mathrm{d}z = 0$.

Fresnelintegralene $\int_0^\infty \cos(t^2)\,\mathrm{d}t$ og $\int_0^\infty \sin(t^2)\,\mathrm{d}t$

Funksjonen $f(z) = \mathrm{e}^{iz^2}$ er analytisk for alle $z \in \mathbb{C}$. Ved Cauchys integralteorem er dermed

$$\oint_{C_R} f(z)\,\mathrm{d}z = 0.$$

Det gir at

$$\begin{aligned}
0 &= \int_0^R f(t)\,\mathrm{d}t + \int_{\Gamma_R} f(z)\,\mathrm{d}z + \int_K f(z)\,\mathrm{d}z \\
&= \int_0^R \mathrm{e}^{it^2}\,\mathrm{d}t + \int_{\Gamma_R} f(z)\,\mathrm{d}z + \int_0^R \mathrm{e}^{i\left((R-t)\mathrm{e}^{i\pi/4}\right)^2}(-\mathrm{e}^{i\pi/4})\,\mathrm{d}t \\
&= \int_0^R \mathrm{e}^{it^2}\,\mathrm{d}t + \int_{\Gamma_R} f(z)\,\mathrm{d}z - \mathrm{e}^{i\pi/4}\int_0^R \mathrm{e}^{-(R-t)^2}\,\mathrm{d}t \\
&= \int_0^R \mathrm{e}^{it^2}\,\mathrm{d}t + \int_{\Gamma_R} f(z)\,\mathrm{d}z - \mathrm{e}^{i\pi/4}\int_0^R \mathrm{e}^{-t^2}\,\mathrm{d}t
\end{aligned}$$

Fresnelintegralene forts.

Dermed er

$$\int_0^R \mathrm{e}^{it^2}\,\mathrm{d}t = \frac{1+i}{\sqrt{2}}\int_0^R \mathrm{e}^{-t^2}\,\mathrm{d}t - \int_{\Gamma_R} \mathrm{e}^{iz^2}\,\mathrm{d}z$$

som gir at

$$\int_0^\infty \mathrm{e}^{it^2}\,\mathrm{d}t = \frac{1+i}{\sqrt{2}}\int_0^\infty \mathrm{e}^{-t^2}\,\mathrm{d}t = \frac{\sqrt{\pi}}{2\sqrt{2}}(1+i).$$

Dessuten er

$$\begin{aligned}\int_0^\infty \mathrm{e}^{it^2}\,\mathrm{d}t &= \int_0^\infty (\cos(t^2) + i\sin(t^2))\,\mathrm{d}t \\ &= \int_0^\infty \cos(t^2)\,\mathrm{d}t + i\int_0^\infty \sin(t^2)\,\mathrm{d}t\end{aligned}$$

så

$$\int_0^\infty \cos(t^2)\,\mathrm{d}t = \int_0^\infty \sin(t^2)\,\mathrm{d}t = \frac{\sqrt{\pi}}{2\sqrt{2}}.$$