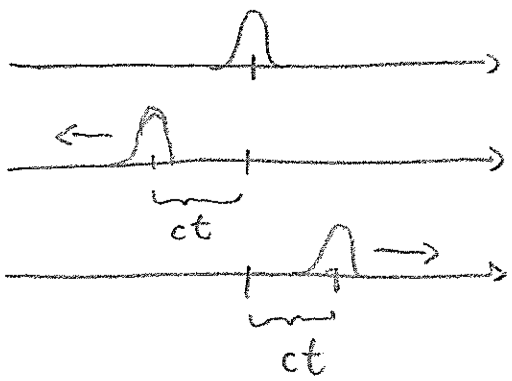
Tolkning: av (5): (når $g \equiv 0$)

$f(x)$ — init. profil

$f(x+ct)$ — bølge mot venstre

$f(x-ct)$ — bølge mot høyre



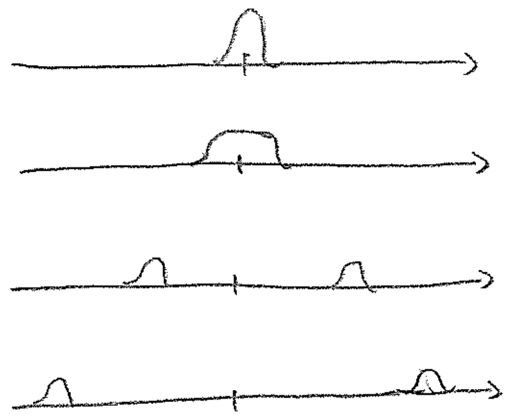
$$u = \frac{1}{2}\left[f(x+ct) + f(x-ct) \right]$$

$t = 0$

$t_1 > 0$

$t_2 > t_1$

$t_3 > t_2$



B. Varmelikn. og F-transform

u temperaturfordelingen i veldig lang ($\approx \infty$ lang) stav m. starttemp. $f(x)$. Løser:

Initialverdiprobl.:

$$\begin{cases} u_t = c^2 u_{xx}, & t > 0,\ x \in \mathbb{R} \quad \text{(varmelikn.)} \\ u(0,x) = f(x), & x \in \mathbb{R} \quad \text{(init. bet.)} \end{cases}$$

Løsn. vha. $\mathcal{F}$-transf.:

1.) $\mathcal{F}$-transf. likn. i x (ikke i t):

$$\mathcal{F}(u_t) = \mathcal{F}(c^2 u_{xx}) = c^2 \mathcal{F}(u_{xx})$$

$$= -c^2 w^2 \mathcal{F}(u) = -c^2 w^2 \hat{u}$$

$$\mathcal{F}(u_t) = \frac{\partial}{\partial t} \mathcal{F}(u) = \hat{u}_t$$

Må vises!
Ikke pensum

Dvs.

$$(*) \begin{cases} \hat{u}_t = -c^2 w^2 \hat{u} & , t > 0 \\ \hat{u}(0, w) = \hat{f}(w) & , t = 0 \end{cases}$$

2.) Bestem $\hat{u}$:

$(*)$ separabel diff. likn.

$\Downarrow$ Mat 3 (sjekk!)

$$\hat{u}(t, w) = \underbrace{\hat{f}(w)\, e^{-c^2 w^2 t}}_{= \hat{f} \cdot \hat{g}, \quad \hat{g} = e^{-c^2 w^2 t}}$$

3.) $\mathcal{F}^{-1}$-transf.:

$$u(x,t) = \mathcal{F}^{-1}(\hat{u}) = \mathcal{F}^{-1}(\hat{f} \cdot \hat{g})$$

$$\mathcal{F}(f * g) = \sqrt{2\pi}\, \hat{f} \cdot \hat{g}$$

$$= \frac{1}{\sqrt{2\pi}} (f * g)(x, t)$$

Best. g fra

$$\mathcal{F}\left(e^{-ax^2}\right) = \frac{1}{\sqrt{2a}}\, e^{-\frac{w^2}{4a}}$$

Velg $a = \frac{1}{4c^2t}$, dvs $\frac{w^2}{4a} = c^2 w^2 t$ og

$$g = \mathcal{F}^{-1}\left(e^{-\frac{w^2}{4a}}\right) = \sqrt{2a}\, e^{-ax^2} \overset{a=\frac{1}{4c^2t}}{=} \underline{\frac{1}{c\sqrt{2t}}\, e^{-\frac{x^2}{4c^2t}}}$$

Konklusjon:

$$\boxed{u(t,x)} = \frac{1}{\sqrt{2\pi}}(f*g)(t,x) = \boxed{\frac{1}{2c\sqrt{\pi t}}\int_{-\infty}^{\infty} f(y)\, e^{-\frac{(x-y)^2}{4c^2t}}\,dy}$$

<u>Eks.:</u>

$$f(x) = \begin{cases} a & -1 < x < 1 \\ 0 & \text{ellers} \end{cases}$$

$$\Rightarrow \quad u(t,x) = \frac{a}{2c\sqrt{\pi t}} \int_{-1}^{1} e^{-\overbrace{\left(\frac{(x-y)^2}{4c^2t}\right)}^{z^2}}\,dy \underset{\text{Sjekk!}}{\overset{z=\sqrt{\frac{(x-y)^2}{4c^2t}}}{=}} \frac{a}{\sqrt{\pi}} \int_{-(1+x)\frac{1}{2c\sqrt{t}}}^{(1-x)\cdot\frac{1}{2c\sqrt{t}}} e^{-z^2}\,dz$$