Solutions of 2nd order linear homogenenous ODEs (informal note)

In this note we consider the following linear homogenenous ODE:

$$y''(x) + ay'(x) + by(x) = 0, \tag{1}$$

where $'$ denotes derivative $\frac{d}{dx}$ and a, b are real constants.

Definition: A general solution of (1) is function $y(x)$ of the form

$$y(x) = C_1 y_1(x) + C_2 y_2(x), \tag{2}$$

where y_1, y_2 are two linearly independent solutions of (1) and C_1, C_2 are (possibly complex) constants.

Remark:

- For given y_1, y_2 *any* solution of (1) can be obtained from (2) by a proper choice of C_1, C_2.
- For the definition of linearly independence and more discussion of general solution, we refer to 9th edition of Kreyszig page 49.

Special solutions of the form $e^{\lambda x}$:

We set $y(x) = e^{\lambda x}$ and insert into (1):

$$(1) \quad \Longleftrightarrow \quad (\lambda^2 + a\lambda + b)e^{\lambda x} = 0 \quad \Longleftrightarrow \quad \lambda = \frac{-a \pm \sqrt{a^2 - 4b}}{2}. \tag{3}$$

Let λ_1 and λ_2 denote the two solutions of (3). We have three different cases:

1	$a^2 - 4b > 0$	$\lambda_1, \lambda_2 \in \mathbb{R}$, $\lambda_1 \neq \lambda_2$
2	$a^2 - 4b = 0$	$\lambda_1 = \lambda_2 = -\frac{a}{2} \in \mathbb{R}$
3	$a^2 - 4b < 0$	$\lambda_1 = -\frac{a}{2} + i\omega$ and $\lambda_1 = -\frac{a}{2} - i\omega$ and $\omega = \sqrt{4b - a^2}$

and in each case the corresponding solutions of (1) are

$$y_1(x) = e^{\lambda_1 x} \quad \text{and} \quad y_2(x) = e^{\lambda_2 x}.$$

The general solution

Remark:

- If $\lambda_1 \neq \lambda_2$ then y_1 and y_2 are linearly independent (check!).
- When $\lambda_1 = \lambda_2 = -\frac{a}{2}$, then $\bar{y}_2(x) := x y_1(x)$ also solve (1) (check!) and is linearly independent of y_1 (check!).
- When $\lambda_1 = -\frac{a}{2} + i\omega$ and $\lambda_1 = -\frac{a}{2} - i\omega$,

$$\tilde{y}_1(x) := \frac{y_1 + y_2}{2} = e^{-\frac{a}{2}x} \cos \omega x \qquad \text{and} \qquad \tilde{y}_2(x) := \frac{y_1 - y_2}{2i} = e^{-\frac{a}{2}x} \sin \omega x$$

also solve (1) by superposition (check!), and $\tilde{y}_1$ and $\tilde{y}_2$ are linearly independent (check!).

By this remark and the definition of general solution, we can now write down the general solution of (1). There are 3 different cases depending on the sign of $a^2 - 4b$:

General solution of (1):

1	$a^2 - 4b > 0$	$y(x) = C_1 y_1(x) + C_2 y_2(x) = C_1 e^{\lambda_1 x} + C_2 e^{\lambda_2 x}$
2	$a^2 - 4b = 0$	$y(x) = C_1 y_1(x) + C_2 \bar{y}_2(x) = (C_1 + C_2 x) e^{\lambda_1 x}$
3	$a^2 - 4b < 0$	$y(x) = C_1 \tilde{y}_1(x) + C_2 \tilde{y}_2(x) = e^{-\frac{a}{2}x}(C_1 \cos \omega x + C_2 \sin \omega x)$

Concluding remarks:

- The quadratic equation found in (3) is called the characteristic equation: $\lambda^2 + a\lambda + b = 0$.
- A more detailed discussion can be found in chapter 2.2 in the textbook "Advanced Engeneering Mathematics" (9th edition) by E. Kreyszig.