Pointwise convergence of Fourier series (informal note)

Let \( f(x) \) be a \( 2\pi \) periodic function where \( \int_{-\pi}^{\pi} |f(x)|dx < \infty \).

Complex Fourier series of \( f(x) \):

\[
\sum_{k=-\infty}^{\infty} c_k e^{ikx}, \quad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx}dx.
\]

Partial sums:

\[
S_{m,n}(x) = \sum_{k=-m}^{n} c_k e^{ikx}.
\]

Complex Bessel inequality:

\[
\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^2 dx.
\]

Proof: \( 0 \leq \int_{-\pi}^{\pi} |S_{m,n} - f|^2 dx = \int_{-\pi}^{\pi} (|S_{m,n}|^2 - 2 \text{Re}[S_{m,n} f] + f^2) dx = -2\pi \sum_{m=-\infty}^{n} |c_k|^2 + \int_{-\pi}^{\pi} f^2 dx \)

Riemann-Lebesgue lemma: \( \lim_{|k| \to \infty} c_k = 0 \).

Proof when \( \int_{-\pi}^{\pi} f(x)^2 dx < \infty \): \( \sum_{k=-\infty}^{\infty} |c_k|^2 < \infty \) by Bessel \( \Rightarrow c_{|k|} \to 0 \) by Divergence test

THEOREM: If \( \int_{-\pi}^{\pi} |f(x)|dx < \infty \) and \( f'(a) \) exists, then \( \lim_{n,m \to \infty} S_{m,n}(a) = f(a) \).
Proof of THEOREM

1. Assume \( a = 0 \) and \( f(a) = f(0) = 0 \). Note that \( f'(0) \) exists by assumption and

\[
\bar{f}(x) := \frac{f(x)}{e^{ix} - 1}
\]

is bounded near 0 and

\[
\int_{-\pi}^{\pi} |\bar{f}(x)| \, dx < \infty.
\]

2. Since \( f(x) = (e^{ix} - 1)\bar{f}(x) \), it follows that

\[
c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)e^{-ikx} \, dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{ix} - 1)\bar{f}(x)e^{-ikx} \, dx = \bar{c}_{k-1} - \bar{c}_k,
\]

and hence

\[
S_{m,n}(0) = \sum_{k=-m}^{n} c_k e^{ik \cdot 0} = \sum_{k=-m}^{n} c_k = \sum_{k=-m}^{n} (\bar{c}_{k-1} - \bar{c}_k) = \bar{c}_{-m-1} - \bar{c}_n.
\]

3. By Riemann-Lebesgue, \( \lim_{n,m \to \infty} S_{m,n}(0) = 0 = f(0) \).

4. Let \( a, f(a) \) be any pair of real numbers, and define

\[
g(x) = f(x + a) - f(a).
\]

Then obviously \( g(0) = 0 \), \( g'(0) \) exists, and \( S_{m,n}^g(x) = S_{m,n}^f(x + a) - f(a) \). Hence, by 3.

\[
|S_{m,n}^f(a) - f(a)| = |S_{m,n}^g(0)| \to 0 \text{ as } m, n \to \infty.
\]

NOTE: The proof is based on the following article,

P. R. Chernoff: Pointwise convergence of Fourier Series.


Further results (for discontinuous functions) and a discussion can be found here.