Pointwise convergence of Fourier series (informal note)

Let $f(x)$ be a 2π periodic function where $\int_{-\pi}^{\pi} |f(x)|dx < \infty$.

Complex Fourier series of $f(x)$:

$$\sum_{k=-\infty}^{\infty} c_k e^{ikx}, \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx.$$

Partial sums:

$$S_{m,n}(x) = \sum_{k=-m}^{n} c_k e^{ikx}.$$

Complex Bessel inequality:

$$\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)^2 dx.$$

Proof: $0 \leq \int_{-\pi}^{\pi} |S_{m,n} - f|^2 dx = \int_{-\pi}^{\pi}(|S_{m,n}|^2 - 2\,\mathrm{Re}[S_{m,n}f] + f^2)dx = -2\pi\sum_{-m}^{n}|c_k|^2 + \int_{-\pi}^{\pi} f^2 dx$

Riemann-Lebesgue lemma: $\lim_{|k|\to\infty} c_k = 0$.

Proof when $\int_{-\pi}^{\pi} f(x)^2 dx < \infty$: $\sum_{-\infty}^{\infty}|c_k|^2 < \infty$ by Bessel $\Rightarrow c_{|k|} \to 0$ by Divergence test

THEOREM: If $\int_{-\pi}^{\pi} |f(x)|dx < \infty$ and $f'(a)$ exists, then $\lim_{n,m\to\infty} S_{m,n}(a) = f(a)$.

Proof of THEOREM

1. Assume $a = 0$ and $f(a) = f(0) = 0$. Note that $f'(0)$ exists by assuption and

$$\bar{f}(x) := \frac{f(x)}{e^{ix} - 1} \qquad \text{is bounded near 0 and} \qquad \int_{-\pi}^{\pi} |\bar{f}(x)| dx < \infty.$$

2. Since $f(x) = (e^{ix} - 1)\bar{f}(x)$, it follows that

$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{ix} - 1)\bar{f}(x) e^{-ikx} dx = \bar{c}_{k-1} - \bar{c}_k,$$

   and hence

$$S_{m,n}(0) = \sum_{k=-m}^{n} c_k e^{ik \cdot 0} = \sum_{k=-m}^{n} c_k = \sum_{k=-m}^{n} (\bar{c}_{k-1} - \bar{c}_k) = \bar{c}_{-m-1} - \bar{c}_n.$$

3. By Riemann-Lebesgue, $\lim_{n,m \to \infty} S_{m,n}(0) = 0 = f(0)$.
4. Let $a, f(a)$ be any pair of real numbers, and define

$$g(x) = f(x + a) - f(a).$$

   Then obviously $g(0) = 0$, $g'(0)$ exists, and $S^g_{m,n}(x) = S^f_{m,n}(x + a) - f(a)$. Hence, by 3.

$$|S^f_{m,n}(a) - f(a)| = |S^g_{m,n}(0)| \to 0 \text{ as } m, n \to \infty.$$

NOTE: The proof is based on the following article,

P. R. Chernoff: Pointwise convergence of Fourier Series.
Amer. Math. Monthly 87 (1980), no. 5, 399–400.

Further results (for discontinuous functions) and a discussion can be found here.