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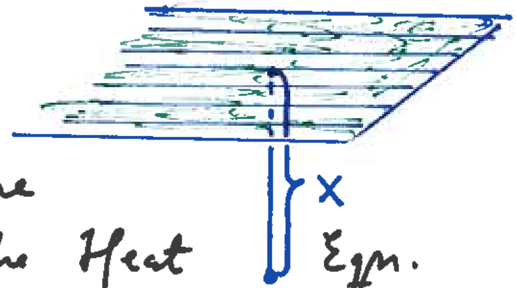
THE PROBLEM OF THE EARTH'S TEMPERATURE



Treating the surface of the earth as a plane, we determine the temperature at depth $x > 0$ from the Heat Eqn.

$$\boxed{\frac{\partial u}{\partial t} = k \frac{\partial^2 u}{\partial x^2}} \qquad (x > 0)$$

We have assumed that $u(x,y,z,t) = u(x,t)$ so that $\underset{\text{LAPLACE}}{\Delta u} = u_{xx}$. At the surface $x = 0$ we have the boundary condition

$$u(0,t) = A\cos\left(\frac{2\pi t}{D}\right) + B\cos\left(\frac{2\pi t}{Y}\right) + C$$

due to daily and yearly variations; here $Y \approx 365\,D$.

Let us first solve the equation with the surface temperature

$$u(0,t) = e^{i\omega t} = \cos(\omega t) + i\sin(\omega t)$$

Then the <u>real part</u> is the desired quantity.

<u>Ansatz</u>: $u(x,t) = X(x)\,T(t)$

$$XT' = X''T\,k$$

$$\frac{T'}{kT} = \frac{X''}{X} = \lambda \quad \text{(const. of separation)}$$

We obtain

$$u(x,t) = c\,e^{kt\lambda}\,e^{\sqrt{\lambda}\,x}$$

$$u(0,t) = c\,e^{kt\lambda} \overset{?}{=} e^{i\omega t}$$

calculate, please!

Thus $c = 1$, $k\lambda = i\omega$.

$$u(x,t) = \underline{e^{i\omega t}\,e^{\sqrt{\frac{i\omega}{k}}\,x}}$$

Now use

$$\sqrt{i} = \pm\frac{1+i}{\sqrt{2}}.$$

It follows that (take $\omega > 0$)

$$u(x,t) = e^{i\omega t}\,e^{\pm(1+i)\sqrt{\frac{\omega}{2k}}\,x}$$

$$= e^{i\left(\omega t \pm \sqrt{\frac{\omega}{2k}}\,x\right)}\,e^{\pm\sqrt{\frac{\omega}{2k}}\,x}$$

The real part is

$$\cos\left(\omega t - \sqrt{\frac{\omega}{2k}}\,x\right)e^{-\sqrt{\frac{\omega}{2k}}\,x},$$

where we have discarded the $+$ sign, since it leads to an unbounded temperature as $x \longrightarrow +\infty$.

<u>Superposition</u> yields the final solution:

$$u(x,t) = A e^{-\sqrt{\frac{\pi}{kD}}\,x} \cos\left(\frac{2\pi}{D}t - \sqrt{\frac{\pi}{kD}}\,x\right)$$
$$+ B e^{-\sqrt{\frac{\pi}{kY}}\,x} \cos\left(\frac{2\pi}{Y}t - \sqrt{\frac{\pi}{kY}}\,x\right)$$
$$+ C.$$

Phase lag
$$\sqrt{\frac{\pi}{kY}}\,x$$

The yearly term has the phase lag
$$\sqrt{\frac{\pi}{kY}}\,x$$
at depth x. At depth x coming from
$$\sqrt{\frac{\pi}{kY}}\,x = \pi, \quad x = \sqrt{\pi k Y}$$
the temperature is totally out of phase from that on the surface: $\cos\left(\frac{2\pi}{Y}t - \pi\right) = -\cos\left(\frac{2\pi}{Y}t\right)$.

MINUS

Choosing a reasonable value of k, leads to a prediction that annual temperature changes will lag by six months at about 2-3 meters depth. — It is winter at a depth of 2-3 meters when it is summer at the surface. (The amplitude is only a fraction of the surface amplitude.)

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