SOLUTIONS TO THE EXAM IN TMA4120/MA2105 H2008

Exercise 1. (i) Let $z = x + iy$. Then

$$z\,\mathrm{Re}(z) = x^2 + iyx.$$

Let $u = x^2$ and $v = yx$. Then

$$u_x = 2x \neq x = v_y, \qquad u_y = 0 \neq -y = -v_x$$

near $z = 1$, and hence the Cauchy-Riemann equations are not satisfied. Therefore, $z\,\mathrm{Re}(z)$ is not analytic at $z = 1$.

(ii) $(z^2)' = 2z$ and z^2 is differentiable in $\mathbb{C}$ and hence analytic at $z = 1$.

(iii) $(\frac{1}{z})' = -\frac{1}{z^2}$ and $\frac{1}{z}$ is differentiable in $\mathbb{C} \setminus \{0\}$ and hence analytic at $z = 1$.

Exercise 2. Taking the Laplace transform of the differential equation gives

$$(s^2 Y - y'(0) - sy(0)) + 4(sY - y(0)) + 4Y = \mathscr{L}(2e^{-2t} + \delta(t-1)),$$

and hence using the table and Rottmann,

$$(s^2 + 4s + 4)Y = (s+2)^2 Y = 2\frac{1}{s+2} + e^{-s}.$$

Therefore,

$$Y = 2\frac{1}{(s+2)^3} + \frac{1}{(s+2)^2}e^{-s}.$$

Taking the inverse Laplace transform

$$\begin{aligned}
\mathscr{L}^{-1}\left(2\frac{1}{(s+2)^3}\right) &= e^{-2t}\mathscr{L}^{-1}\left(\frac{2}{s^3}\right) = e^{-2t}t^2, \\
\mathscr{L}^{-1}\left(\frac{1}{(s+2)^2}e^{-s}\right) &= \mathscr{L}^{-1}\left(\frac{1}{(s+2)^2}\right)\Big|_{t=t-1} u(t-1) \\
&= \left[e^{-2t}t\right]_{t=t-1} u(t-1) = e^{-2(t-1)}(t-1)u(t-1),
\end{aligned}$$

and therefore,

$$y(t) = \underline{t^2 e^{-2t} + (t-1)e^{-2(t-1)}u(t-1)}.$$

Exercise 3. Note that f is an even function with period $p = 2L = 4 \Rightarrow L = 2$. The Fourier coefficients are calculated as follows:

$$\begin{aligned}
a_0 &= \frac{1}{2L}\int_{-L}^{L} f(x)dx = 2\frac{1}{2\cdot 2}\int_0^2 (1-x)dx = \underline{0},\\
a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)dx = \int_0^2 (1-x)\cos\left(\frac{n\pi x}{2}\right)dx\\
&= \left[\frac{2}{n\pi}(1-x)\sin\left(\frac{n\pi x}{2}\right)\right]_0^2 - \frac{2}{n\pi}\int_0^2 (-1)\sin\left(\frac{n\pi x}{2}\right)dx\\
&= (\frac{2}{n\pi})^2\left[\cos\left(\frac{n\pi x}{2}\right)\right]_0^2 = (\frac{2}{n\pi})^2(1-\cos(n\pi))\\
&= (\frac{2}{n\pi})^2(1-(-1)^n) = \underline{\begin{cases}\frac{8}{\pi^2}\frac{1}{(2m+1)^2}, & n = 2m+1\\ 0, & n = 2m,\end{cases}}\\
b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)dx = \underline{0}.
\end{aligned}$$

The Fourier series of f is therefore

$$\frac{8}{\pi^2}\sum_0^\infty \frac{\cos\left(\frac{2m+1}{2}\pi x\right)}{(2m+1)^2} = \frac{8}{\pi^2}\left(\cos\left(\frac{\pi x}{2}\right) + \frac{1}{3^2}\cos\left(\frac{3\pi x}{2}\right) + \dots\right).$$

Exercise 4. (a) Let $f(z) = 3z^2 + 10iz - 3$.

(i) The singular points of f are the zeros of the denominator:

$$3z^2 + 10iz - 3 = 0 \Rightarrow z = \frac{-10i \pm \sqrt{-100+36}}{6} = \frac{-10\pm 8}{6}i.$$

Therefore, f has poles of order one at $z = -3i$ and $z = -\frac{1}{3}i$.

(ii) The singularity located inside C: $z = -\frac{1}{3}i$.

(iii) Residue of f at $z = -\frac{1}{3}i$:

$$\operatorname*{Res}_{z=-\frac{1}{3}i} f(z) = \frac{1}{(3z^2+10iz-3)'}\Big|_{z=-\frac{1}{3}i} = \frac{1}{6z+10i}\Big|_{z=-\frac{1}{3}i} = -\frac{i}{8}.$$

(iv) By the Residue Theorem:

$$\oint_C f(z)dz = 2\pi i \operatorname*{Res}_{z=-\frac{1}{3}i} f(z) = \underline{\frac{\pi}{4}}.$$

(b) Substituting $t = 2x$ we get

$$I = \int_0^\pi \frac{2}{3\sin(2x)+5}dx = \int_0^{2\pi}\frac{1}{3\sin t + 5}dt.$$

We use the complex substitution:

$$z = e^{it}, \quad dz = ie^{it}dt = izdt.$$

Note that $C = \{e^{it} :\ t \in [0, 2\pi]\}$ is the whole unit circle $|z| = 1$, with counterclockwise orientation, and that

$$\sin t = \frac{1}{2i}\left(e^{it} - e^{-it}\right) = \frac{1}{2i}\left(z - \frac{1}{z}\right).$$

These considerations and (a) then gives

$$I = \oint_C \frac{1}{\frac{3}{2i}\left(z - \frac{1}{z}\right) + 5} \frac{dz}{iz} = 2\oint_C \frac{1}{3z^2 + 10iz - 3} dz = 2\frac{\pi}{4} = \underline{\frac{\pi}{2}}.$$

Exercise 5. (a) Set $u(x,t) = F(x)G(t)$ in (1) and (2):

$$(4) \quad FG' - F'G = F'''G, \quad t \geq 0, \quad 0 \leq x \leq \frac{2\pi}{\sqrt{7}}$$

$$(5) \quad F(0)G(t) = 0 = F(\frac{2\pi}{\sqrt{7}})G(t), \quad t \geq 0$$

Multiply (4) with $\frac{1}{FG}$:

$$\frac{G'(t)}{G(t)} = \frac{F'(x) + F'''(x)}{F(x)} = const. = k \quad \Rightarrow \quad \underline{G' = kG} \quad \text{and} \quad \underline{F''' + F' = kF}.$$

Equation (5) implies that either $G \equiv 0$ ($\Rightarrow u \equiv 0$) or

$$\underline{F(0) = 0 = F(\frac{2\pi}{\sqrt{7}})}.$$

We solve for G:

$$G' = kG \Rightarrow \underline{G = Ce^{kt}}.$$

(b) From (3) we have

$$2e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{7}}{2}x\right) = F(x)G(0) = F(x)C \quad \Rightarrow \quad F(x) = \frac{2}{C}e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{7}}{2}x\right).$$

We determine k from the equation $F''' + F' = kF$ as follows:

$$\begin{aligned}
F' &= -\frac{1}{2}F + \frac{\sqrt{7}}{C}e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{7}}{2}x\right) \\
F'' &= \frac{1}{4}F - \frac{\sqrt{7}}{C}e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{7}}{2}x\right) - \frac{7}{4}\frac{2}{C}e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{7}}{2}x\right) \\
&= -\frac{3}{2}F - \frac{\sqrt{7}}{C}e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{7}}{2}x\right), \\
F''' &= \frac{3}{4}F - \frac{3}{2}\frac{\sqrt{7}}{C}e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{7}}{2}x\right) + \frac{1}{2}\frac{\sqrt{7}}{C}e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{7}}{2}x\right) + \frac{7}{4}F \\
&= \frac{5}{2}F - \frac{\sqrt{7}}{C}e^{-\frac{x}{2}} \cos\left(\frac{\sqrt{7}}{2}x\right).
\end{aligned}$$

Therefore,

$$F''' + F' = (\frac{5}{2} - \frac{1}{2})F + 0 = 2F,$$

and hence $k = 2$. The solution of (1), (2), and (3) then becomes

$$u(x,t) = F(x)G(t) = \frac{2}{C}e^{-\frac{x}{2}} \sin\left(\frac{\sqrt{7}}{2}x\right) Ce^{2t} = \underline{2e^{2t-\frac{x}{2}} \sin\left(\frac{\sqrt{7}}{2}x\right)}.$$

Exercise 6. The residue is coefficients in front of $\frac{1}{z-1}$-term in the Laurentz series of f with center at $z=1$.

(1) Taylor series of $e^{\frac{1}{z}}$, center at $z=1$: $e^{\frac{1}{z}}$ is analytic in $\mathbb{C}\setminus\{0\}$, hence Taylors Theorem gives

$$e^{\frac{1}{z}} = \sum_{n=0}^{\infty} a_n (z-1)^n, \qquad |z-1|<1, \quad a_n = \frac{1}{n!}\left(e^{\frac{1}{z}}\right)^{(n)}\Big|_{z=1}.$$

(2) Laurent series of $\frac{e^{\frac{1}{z}}}{(z-1)^2}$:

$$\frac{e^{\frac{1}{z}}}{(z-1)^2} = \sum_{n=0}^{\infty} a_n (z-1)^{n-2} = \frac{a_0}{(z-1)^2} + \frac{a_1}{z-1} + \sum_{n=0}^{\infty} a_{n+2}(z-1)^n.$$

The series converges for $z \neq 1$ or equivalently $|z-1|>0$.

(3) Laurent series of $e^{\frac{1}{1-z}}$: Substituting $u=\frac{1}{1-z}$,

$$e^{\frac{1}{1-z}} = e^u = \sum_{n=0}^{\infty} \frac{u^n}{n!} = \sum_{n=0}^{\infty} \frac{1}{n!(1-z)^n} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{1}{(z-1)^n}.$$

The series converges for $|u|<\infty \Rightarrow |z-1|>0$.

(4) Laurent series of $(z-1)^2 e^{\frac{1}{1-z}}$:

$$\begin{aligned}(z-1)^2 e^{\frac{1}{1-z}} &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}\frac{1}{(z-1)^{n-2}} \\ &= (z-1)^2 - (z-1) + \frac{1}{2} - \frac{1}{3!}\frac{1}{z-1} + \sum_{n=4}^{\infty} \frac{(-1)^n}{n!}\frac{1}{(z-1)^{n-2}}.\end{aligned}$$

The series converges for $|z-1|>0$.

(5) Laurent series of $f(z)$:

$$\begin{aligned}f(z) = \sum_{n=0}^{\infty} a_{n+2}(z-1)^n &+ \frac{a_0}{(z-1)^2} + \frac{a_1}{z-1} \\ &+ (z-1)^2 - (z-1) + \frac{1}{2} - \frac{1}{3!}\frac{1}{z-1} + \sum_{n=4}^{\infty} \frac{(-1)^n}{n!}\frac{1}{(z-1)^{n-2}}.\end{aligned}$$

The series converges for $0<|z-1|<1$.

(6) By definition,

$$\operatorname*{Res}_{z=1} f(z) = a_1 - \frac{1}{3!} = \frac{1}{1!}\left(e^{\frac{1}{z}}\right)'\Big|_{z=1} - \frac{1}{6} = \underline{-e - \frac{1}{6}}.$$