Laplace transform

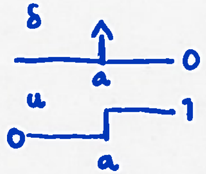
- $F(s) = \mathcal{L}(f(t)) = \int_0^\infty e^{-st} f(t)\,dt$

Regne-regler

- $\mathcal{L}[af(t) + bg(t)] = a\mathcal{L}(f) + b\mathcal{L}(g)$
- $\boxed{\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)}$
- $\mathcal{L}[f''] = s^2\mathcal{L}[f] - sf(0) - f'(0)$
- $\mathcal{L}[e^{at}f(t)] = F(s-a)$
- $\mathcal{L}[f(t-a)u(t-a)] = e^{-as}F(s)$
- $\mathcal{L}[\underbrace{f*g}_{=\int_0^t f(t-s)g(s)\,ds}] = \mathcal{L}(f)\cdot\mathcal{L}(g)$

δ-funk: $\mathcal{L}[\delta(t-a)] = e^{-as}$

Heaviside: $\mathcal{L}[u(t-a)] = \frac{e^{-as}}{s}$



Anvendelse

$y'' + ay' + by = f \;,\; y'(0) = c,\; y(0) = d$

$(s^2 + as + b)Y = F + \cdots$

$Y = \cdots \rightarrow y = \mathcal{L}^{-1}(Y) = \ldots$

Fourier transform

- $\hat{f}(\omega) = \mathcal{F}(f(x)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} f(x)e^{-i\omega x}\,dx$
- $\check{g}(x) = \mathcal{F}^{-1}(g(\omega)) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} g(\omega)e^{i\omega x}\,d\omega$

- $\mathcal{F}[af(x) + bg(x)] = a\mathcal{F}(f) + b\mathcal{F}(g)$
- $\mathcal{F}[f'] = i\omega\mathcal{F}(f)$ $[f(x)\to 0, |x|\to\infty]$
- $\mathcal{F}[f''] = -\omega^2\mathcal{F}(f)$ $[f, f' \to 0, |x|\to\infty]$
- $\mathcal{F}[e^{iax}f(x)] = \hat{f}(\omega - a)$
- $\mathcal{F}[\underbrace{f*g}_{=\int_{-\infty}^{\infty} f(x-y)g(y)\,dy}] = \sqrt{2\pi}\,\mathcal{F}(f)\cdot\mathcal{F}(g)$

Invers: $\mathcal{F}^{-1}\mathcal{F}(f) = f$

Fourierintegral: $\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty}\hat{f}(\omega)e^{i\omega x}\,d\omega$ $(= f(x))$

$u_t = c^2 u_{xx} \;,\; t>0,\; x\in\mathbb{R} \;,\; u|_{t=0} = f$

$\hat{u}_t = -c^2\omega^2\hat{u} \;,\; \hat{u}|_{t=0} = \hat{f}$

$\hat{u} = \cdots \rightarrow u = \mathcal{F}^{-1}(\hat{u}) = \ldots$

Fourierrk:

- 2L periodisk $\boxed{f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right)}$

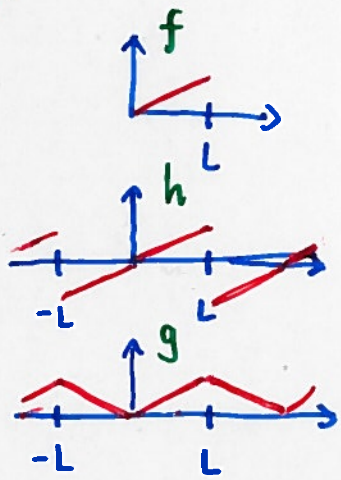
$\boxed{a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)\,dx}$ $\boxed{a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}\,dx}$ $\boxed{b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}\,dx}$

- Konvergens: Rekkens sum $S(x) = \begin{cases} f(x), & f \text{ kont. i } x \\ \frac{1}{2}(f(x^-)+f(x^+)), & f \text{ diskont. i } x \end{cases}$

  $\hookrightarrow$ Når f stk.vis kont. og H. og V. deriv. eks.

- Parseval: $2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{L}\int_{-L}^{L} f(x)^2\,dx$

Utvidelser av $f(x), x \in [0,L]$ til $\mathbb{R}$:



- Odde 2L-periodisk utv. $h(x), x \in \mathbb{R}$:

$h(x) = f(x)$	$x \in [0,L]$	(utvidelse)
$h(-x) = -h(x)$	$x \in \mathbb{R}$	(odde)
$h(x+2L) = h(x)$	$x \in \mathbb{R}$	(periodisk)

- Like 2L-periodisk utv. $g(x), x \in \mathbb{R}$:

Fourier sin rk til $f(x), x \in [0,L]$ = Fourierrk. til $h(x)$

$\boxed{f(x) = \sum_{n=1}^{\infty} a_n \sin\frac{n\pi x}{L}}$ $\quad a_0 = 0 = a_n$ (odde integrand) $\quad \boxed{b_n = \frac{1}{L}\int_{-L}^{L} \underbrace{h \sin\frac{n\pi x}{L}}_{\text{like}}\,dx = \frac{2}{L}\int_0^L f \sin\frac{n\pi x}{L}\,dx}$

Fourier cos rk til $f(x), x \in [0,L]$ = Fourierrk. til $g(x)$

$\boxed{f(x) = a_0 + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}}$ $\boxed{a_0 = \frac{1}{L}\int_0^L f(x)\,dx}$ $\boxed{a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}\,dx}$

Partielle diff. likn. (PDL):

- Bølgelikn. $u_{tt} = c^2 u_{xx}$, $(t,x) \in D$ hyperbolsk
- Varmelikn. $u_t = c^2 u_{xx}$; $(t,x) \in D$ parabolsk
- Laplace likn. $u_{xx} + u_{yy} = 0$, $(x,y) \in D$ elliptisk

Løsning:

- Tilstrekkelig deriverbar funk. u som oppfyller PDL i alle pkt. i D
- Løsn. er ikke entydig uten tilleggsbetingelser:
  1.) Cauchy bet. = initial bet.: Eks. $u(x, t=0) = f(x)$
  2.) Dirichlet randbet.: Eks. $u(x=0,t) = 0$, $u(x=1,t) = 3$
  3.) Neumann randbet.: Eks. $u_x(x=0,t) = 0 = u_x(x=1,t)$
- Eks: Cauchy - Neumann problem

$$\begin{cases} u_t = c^2 u_{xx}, & t>0,\ x\in(0,\pi) \\ u(t=0) = f(x), & x \in (0,\pi) \\ u_x(x=0) = 0 = u_x(x=\pi), & t>0 \end{cases}$$

Løsning:
$u(x,t) = e^{-c^2 t}\cos x$
når $f = \cos x$

Superposisjon:

u, v løser linear homogen PDL $\Rightarrow c_1 u + c_2 v$ løser samme PDL

Løsningsmetoder:

- Fouriertransform $(x \in \mathbb{R})$, Laplacetransform $(x \in [0,\infty))$
- Separasjon av var. $(x \in [0,L])$
  + Fourierrekker
- D'Alembert: Alle løsn. av bølgelikn. er på formen

$$\varphi(x+ct) + \psi(x-ct) \qquad [\text{best. } \varphi, \psi]$$

Separasjon av var

1. PDL + homogene tilleggsbet. (produktløsn'er)

Finn alle løsn. på formen $u(x,t) = F(x) \cdot G(t)$:

$$u_n = F_n \cdot G_n, \quad n = 0, 1, 2, 3, \dots$$

2. Inhomogen tilleggsbet. (superposisjon)

- $u = \sum_{n=0}^{\infty} C_n u_n$, u_n fra **1**
- Velg C_n s.a. u tilfredstiller inhomogen tilleggsbet. (bruk F-rk.

3. Sjekk at u fra **2** løser PDL og hom. tilleggsbet.

Inhomogene problem

$$Au = f \text{ i } \Omega \quad \text{og} \quad u|_{\partial\Omega} = g$$

$$A = a\frac{\partial^2}{\partial x^2} + b\frac{\partial^2}{\partial x \partial y} + c\frac{\partial^2}{\partial y^2}$$

$\boxed{u = u_1 + u_2}$ der

$$\begin{cases} Au_1 = f \text{ i } \Omega, & u_1|_{\partial\Omega} = 0 \\ Au_2 = 0 \text{ i } \Omega, & u_2|_{\partial\Omega} = g \end{cases}$$

"superposisjon"

Anal. funk.:

- Analytisk i z_0 = deriverbar i z_0 (+ liten omegn)
- Anal. $\Rightarrow$ ∞ deriv. + konv. Taylorrk.
- Cauchy-Riemann: $f(z) = u(x,y) + iv(x,y)$ anal. i D

$$\Updownarrow$$

$$\boxed{u_x = v_y,\ u_y = -v_x \text{ i } D}$$ [u_x, u_y, v_x, v_y kont.]

- $f = u + iv$ anal. i D $\Rightarrow$ $\boxed{u_{xx} + u_{yy} = 0}$, $\boxed{v_{xx} + v_{yy} = 0}$ i D

  u og v konjugerte harmoniske funk.

- Cauchys int.-tm.: $\boxed{\oint_C f(z)\,dz = 0}$ for alle enkle lukka C der f anal. på og innenfor C.
- Cauchys int. formel: $f(z) = \frac{1}{2\pi i} \oint_C \frac{f(z^*)}{z - z^*} dz^*$ (— " — C mot kl.)

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Rekker:

- Taylor: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$, $\boxed{|z - z_0| < R}$ $\left[a_n = \frac{f^{(n)}(z_0)}{n!}\right]$
- Laurent: $f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$, $\boxed{r < |z - z_0| < R}$
- Forutsetning: $f(z)$ anal. på $|z - z_0| < R$ / $r < |z - z_0| < R$

Singulariteter:

- Pkt. der $f(z)$ ikke anal./def. ...
- Isolert sing. z_1 = eneste sing. i liten disk

  1) Orden n pol z_0: $f(z) = \sum a_n (z - z_0)^n + \frac{b_1}{z - z_0} + \dots + \frac{b_n}{(z - z_0)^n}$, $0 < |z - z_0| <$

  2) Isa ess. sing z_0: $f(z) = \sum a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}$, $0 < |z - z_0| < R$

  3) Hevbar sing.: ...

Residy:

- $\operatorname*{Res}_{z=z_0} f(z) = b_1$ når $f(z) = a_0 + a_1(z-z_0) + \dots + \frac{b_1}{z-z_0} + \frac{b_2}{z-z_0} + \dots,$ $0<|z-z_0|<R$
- z_0 orden n pol: $\boxed{\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} \frac{1}{(n-1)!}\left((z-z_0)^n f(z)\right)^{(n-1)}}$

Residytm: $\boxed{\oint_C f(z)\,dz = 2\pi i \sum_{j=1}^{m} \operatorname*{Res}_{z=z_j} f(z)}$

- Forutsetning:
  - C enkel, lukka, mot kl.
  - $f(z)$ anal. på og innenfor C utenom iso. sing. $z_1, \dots, z_m$.

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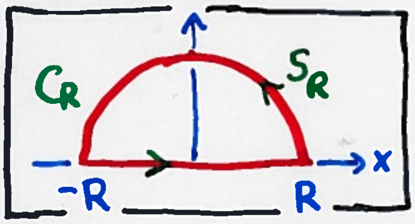
Reelle int:

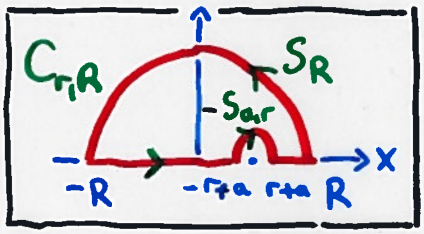
1. $\boxed{\int_0^{2\pi} F(\cos\theta, \sin\theta)\,d\theta} \overset{z=e^{i\theta}}{=} \oint_{|z|=1} F\left(\tfrac{1}{2}(z+\tfrac{1}{z}), \tfrac{1}{2i}(z-\tfrac{1}{z})\right)\frac{dz}{iz}$

2. $\boxed{\int_{-\infty}^{\infty} f(x)\,dx} = \lim_{R\to\infty} \underbrace{\int_{-R}^{R} f(x)\,dx}$



$= \underbrace{\oint_{C_R} f(z)\,dz}_{\text{Residytm.}} - \underbrace{\int_{S_R} f(z)\,dz}_{\xrightarrow[R\to\infty]{\text{ML-ulikh.}} 0}$

3. $\boxed{\text{pr.v.} \int_{-\infty}^{\infty} f(x)\,dx} \overset{f \text{ sing. i } z=a\in\mathbb{R}}{=} \lim_{\substack{R\to\infty \\ r\to 0}} \underbrace{\left(\int_{-R}^{a-r} + \int_{a+r}^{R}\right) f(x)\,dx}$



$= \underbrace{\oint_{C_{r,R}} f(z)\,dz}_{\text{Residytm.}} - \underbrace{\int_{S_R} f(z)\,dz}_{\xrightarrow[R\to\infty]{} 0} - \underbrace{\int_{-S_{r,a}} f(z)\,dz}_{\xrightarrow[r\to 0]{} -\pi i \operatorname*{Res}_{z=a} f(z)}$

4. $\boxed{\int_{-\infty}^{\infty} g(x)\, e^{-i\omega x}\,dx}$ / $\boxed{\text{pr.v.} \int_{-\infty}^{\infty} g(x)\, e^{-i\omega x}\,dx}$

- Som i **2** / **3** m. $f(x) = g(x)\,e^{-i\omega x}$
- Må velge C_R / $C_{r,R}$ i det halvplan der $|e^{-i\omega z}| \leq 1$!