We insert these two results into the formula for a_n. The sine terms cancel and so does a factor L^2. This gives

$$a_n = \frac{4k}{n^2\pi^2}\left(2\cos\frac{n\pi}{2} - \cos n\pi - 1\right).$$

Thus,

$$a_2 = -16k/(2^2\pi^2), \qquad a_6 = -16k/(6^2\pi^2), \qquad a_{10} = -16k/(10^2\pi^2), \cdots$$

and $a_n = 0$ if $n \neq 2, 6, 10, 14, \cdots$. Hence the first half-range expansion of $f(x)$ is (Fig. 272a)

$$f(x) = \frac{k}{2} - \frac{16k}{\pi^2}\left(\frac{1}{2^2}\cos\frac{2\pi}{L}x + \frac{1}{6^2}\cos\frac{6\pi}{L}x + \cdots\right).$$

This Fourier cosine series represents the even periodic extension of the given function $f(x)$, of period $2L$.

(b) *Odd periodic extension.* Similarly, from (6**) we obtain

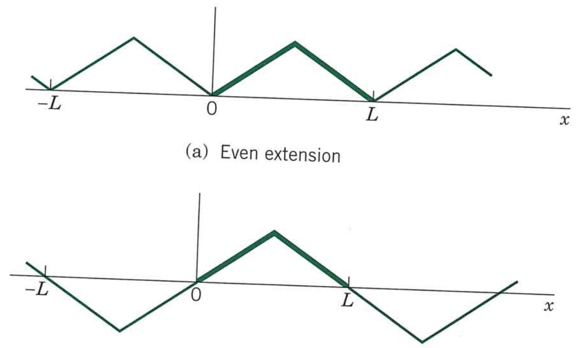
$$b_n = \frac{8k}{n^2\pi^2}\sin\frac{n\pi}{2}. \tag{5}$$

Hence the other half-range expansion of $f(x)$ is (Fig. 272b)

$$f(x) = \frac{8k}{\pi^2}\left(\frac{1}{1^2}\sin\frac{\pi}{L}x - \frac{1}{3^2}\sin\frac{3\pi}{L}x + \frac{1}{5^2}\sin\frac{5\pi}{L}x - + \cdots\right).$$

The series represents the odd periodic extension of $f(x)$, of period $2L$.

Basic applications of these results will be shown in Secs. 12.3 and 12.5. ■



(a) Even extension

(b) Odd extension

Fig. 272. Periodic extensions of $f(x)$ in Example 6

PROBLEM SET 11.2

1–7 EVEN AND ODD FUNCTIONS

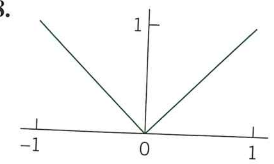
Are the following functions even or odd or neither even nor odd?

1. e^x, $e^{-|x|}$, $x^3\cos nx$, $x^2\tan\pi x$, $\sinh x - \cosh x$
2. $\sin^2 x$, $\sin(x^2)$, $\ln x$, $x/(x^2+1)$, $x\cot x$
3. Sums and products of even functions
4. Sums and products of odd functions
5. Absolute values of odd functions
6. Product of an odd times an even function
7. Find all functions that are both even and odd.

8–17 FOURIER SERIES FOR PERIOD $p = 2L$

Is the given function even or odd or neither even nor odd? Find its Fourier series. Show details of your work.

8.



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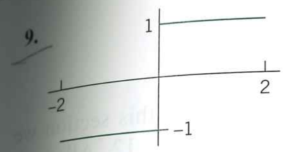

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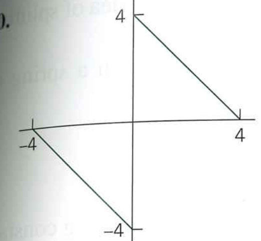
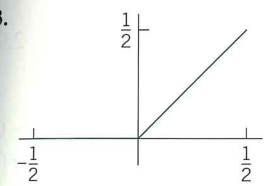
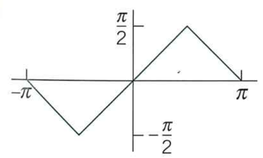


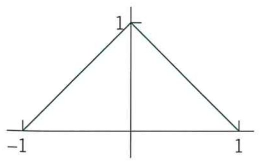
11. $f(x) = x^2 \quad (-1 < x < 1), \quad p = 2$
12. $f(x) = 1 - x^2/4 \quad (-2 < x < 2), \quad p = 4$
13.



14. $f(x) = \cos\pi x \quad (-\frac{1}{2} < x < \frac{1}{2}), \quad p = 1$
15.



16. $f(x) = x|x| \quad (-1 < x < 1), \quad p = 2$
17.



18. **Rectifier.** Find the Fourier series of the function obtained by passing the voltage $v(t) = V_0\cos 100\pi t$ through a half-wave rectifier that clips the negative half-waves.
19. **Trigonometric Identities.** Show that the familiar identities $\cos^3 x = \frac{3}{4}\cos x + \frac{1}{4}\cos 3x$ and $\sin^3 x = \frac{3}{4}\sin x - \frac{1}{4}\sin 3x$ can be interpreted as Fourier series expansions. Develop $\cos^4 x$.
20. **Numeric Values.** Using Prob. 11, show that $1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots = \frac{1}{6}\pi^2$.
21. **CAS PROJECT. Fourier Series of 2L-Periodic Functions. (a)** Write a program for obtaining partial sums of a Fourier series (5).

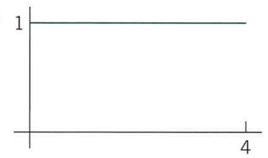
(b) Apply the program to Probs. 8–11, graphi few partial sums of each of the four series o axes. Choose the first five or more partial they approximate the given function reasor Compare and comment.

22. Obtain the Fourier series in Prob. 8 frc Prob. 17.

23–29 HALF-RANGE EXPANSIONS

Find **(a)** the Fourier cosine series, **(b)** the Fourier Sketch $f(x)$ and its two periodic extensions. details.

23.



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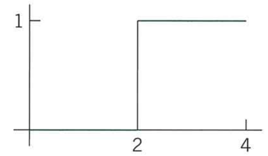

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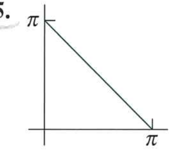

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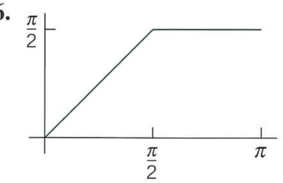

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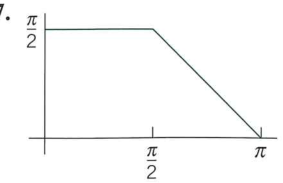

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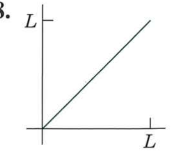

29. $f(x) = \sin x \; (0 < x < \pi)$
30. Obtain the solution to Prob. 26 fron Prob. 27.

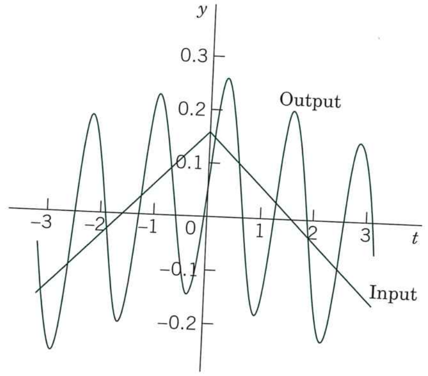

Fig. 277. Input and steady-state output in Example 1

PROBLEM SET 11.3

1. **Coefficients C_n.** Derive the formula for C_n from A_n and B_n.
2. **Change of spring and damping.** In Example 1, what happens to the amplitudes C_n if we take a stiffer spring, say, of $k = 49$? If we increase the damping?
3. **Phase shift.** Explain the role of the B_n's. What happens if we let $c \to 0$?
4. **Differentiation of input.** In Example 1, what happens if we replace $r(t)$ with its derivative, the rectangular wave? What is the ratio of the new C_n to the old ones?
5. **Sign of coefficients.** Some of the A_n in Example 1 are positive, some negative. All B_n are positive. Is this physically understandable?

6–11 GENERAL SOLUTION

Find a general solution of the ODE $y'' + \omega^2 y = r(t)$ with $r(t)$ as given. Show the details of your work.

6. $r(t) = \sin \alpha t + \sin \beta t, \omega^2 \neq \alpha^2, \beta^2$
7. $r(t) = \sin t, \omega = 0.5, 0.9, 1.1, 1.5, 10$
8. **Rectifier.** $r(t) = \pi/4 \, |\cos t|$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t), |\omega| \neq 0, 2, 4, \cdots$
9. What kind of solution is excluded in Prob. 8 by $|\omega| \neq 0, 2, 4, \cdots$?
10. **Rectifier.** $r(t) = \pi/4 \, |\sin t|$ if $0 < t < 2\pi$ and $r(t + 2\pi) = r(t), |\omega| \neq 0, 2, 4, \cdots$
11. $r(t) = \begin{cases} -1 & \text{if} \quad -\pi < t < 0 \\ 1 & \text{if} \quad 0 < t < \pi, \end{cases} \quad |\omega| \neq 1, 3, 5, \cdots$
12. **CAS Program.** Write a program for solving the ODE just considered and for jointly graphing input and output of an initial value problem involving that ODE. Apply the program to Probs. 7 and 11 with initial values of your choice.

13–16 STEADY-STATE DAMPED OSCILLATIONS

Find the steady-state oscillations of $y'' + cy' + y = r(t)$ with $c > 0$ and $r(t)$ as given. Note that the spring constant is $k = 1$. Show the details. In Probs. 14–16 sketch $r(t)$.

13. $r(t) = \sum_{n=1}^{N} (a_n \cos nt + b_n \sin nt)$
14. $r(t) = \begin{cases} -1 & \text{if} \quad -\pi < t < 0 \\ 1 & \text{if} \quad 0 < t < \pi \end{cases}$ and $r(t + 2\pi) = r(t)$
15. $r(t) = t(\pi^2 - t^2)$ if $-\pi < t < \pi$ and $r(t + 2\pi) = r(t)$
16. $r(t) = \begin{cases} t & \text{if } -\pi/2 < t < \pi/2 \\ \pi - t & \text{if } \pi/2 < t < 3\pi/2 \end{cases}$ and $r(t + 2\pi) = r(t)$

17–19 *RLC*-CIRCUIT

Find the steady-state current $I(t)$ in the *RLC*-circuit in Fig. 275, where $R = 10 \, \Omega$, $L = 1$ H, $C = 10^{-1}$ F and with $E(t)$ V as follows and periodic with period 2π. Graph or sketch the first four partial sums. Note that the coefficients of the solution decrease rapidly. *Hint.* Remember that the ODE contains $E'(t)$, not $E(t)$, cf. Sec. 2.9.

17. $E(t) = \begin{cases} -50t^2 & \text{if} \quad -\pi < t < 0 \\ 50t^2 & \text{if} \quad 0 < t < \pi \end{cases}$
18. $E(t) = \begin{cases} 100\,(t - t^2) & \text{if} \quad -\pi < t < 0 \\ 100\,(t + t^2) & \text{if} \quad 0 < t < \pi \end{cases}$
19. $E(t) = 200t(\pi^2 - t^2) \quad (-\pi < t < \pi)$
20. **CAS EXPERIMENT. Maximum Output** Graph and discuss outputs of $y'' + cy' + ky = r$ $r(t)$ as in Example 1 for various c and k with emp the maximum C_n and its ratio to the second large

11.4 Approximation by Trigonometric Polynomials

Fourier series play a prominent role not only in differential equations but **approximation theory**, an area that is concerned with approximating funct other functions—usually simpler functions. Here is how Fourier series come picture.

Let $f(x)$ be a function on the interval $-\pi \leqq x \leqq \pi$ that can be represented interval by a Fourier series. Then the ***N*th partial sum** of the Fourier series

$$f(x) \approx a_0 + \sum_{n=1}^{N} (a_n \cos nx + b_n \sin nx) \tag{1}$$

is an approximation of the given $f(x)$. In (1) we choose an arbitrary N and keep Then we ask whether (1) is the "best" approximation of f by a **trigonometric pol of the same degree N**, that is, by a function of the form

$$F(x) = A_0 + \sum_{n=1}^{N} (A_n \cos nx + B_n \sin nx) \tag{2}$$

Here, "best" means that the "error" of the approximation is as small as possibl

Of course we must first define what we mean by the **error** of such an appro We could choose the maximum of $|f(x) - F(x)|$. But in connection with Fou it is better to choose a definition of error that measures the goodness of agreemen *f* and *F on the whole interval* $-\pi \leqq x \leqq \pi$. This is preferable since the sum f of series may have jumps: F in Fig. 278 is a good overall approximation of f, but the of $|f(x) - F(x)|$ (more precisely, the *supremum*) is large. We choose

$$E = \int_{-\pi}^{\pi} (f - F)^2 \, dx. \tag{3}$$

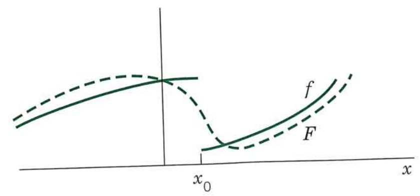

Fig. 278. Error of approximation