Pointwise convergence of Fourier series

Let $f(x)$ be a 2π periodic function and $\int_{-\pi}^{\pi} |f(x)| dx < \infty$.

Complex Fourier: $S_f(x) = \sum_{k=-\infty}^{\infty} c_k e^{ikx}, \quad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx$

Partial sums: $S_{m,n}(x) = \sum_{k=-m}^{n} c_k e^{ikx}$

Bessel inequality: $\sum_{k=-\infty}^{\infty} |c_k|^2 \leq \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x)^2 dx$

Proof:

$0 \leq \int_{-\pi}^{\pi} |S_{m,n} - f|^2 dx = \int_{-\pi}^{\pi} (|S_{m,n}|^2 - 2\,\mathrm{Re}[S_{m,n} f] + f^2) dx = -2\pi \sum_{-m}^{n} |c_k|^2 + \int_{-\pi}^{\pi} f^2 dx$ □

Riemann-Lebesgue lemma: $\lim_{|k| \to \infty} c_k = 0$

Proof when $\int_{-\pi}^{\pi} f(x)^2 dx < \infty$: $\sum_{-\infty}^{\infty} |c_k|^2 < \infty$ by Bessel $\Rightarrow c_{|k|} \to 0$ by Divergence test □

Theorem

If $\int_{-\pi}^{\pi} |f(x)| dx < \infty$ *and* $f'(a)$ *exists, then* $S_f(a) = \lim_{n,m \to \infty} S_{m,n}(a) = f(a)$.

Proof of Theorem

From P.R. Chernoff: Pointwise convergence of Fourier Series. *Amer. Math. Monthly* 87(5): 399–400, 1980

1. Assume $a = 0$ and $f(a) = f(0) = 0$. Note that $f'(0)$ exists by assuption,
$$\bar{f}(x) := \frac{f(x)}{e^{ix} - 1} \quad \text{is bounded near 0,} \qquad \text{and} \qquad \int_{-\pi}^{\pi} |\bar{f}(x)| dx < \infty.$$
2. Since $f(x) = (e^{ix} - 1)\bar{f}(x)$, it follows that
$$c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-ikx} dx = \frac{1}{2\pi} \int_{-\pi}^{\pi} (e^{ix} - 1)\bar{f}(x) e^{-ikx} dx = \bar{c}_{k-1} - \bar{c}_k,$$
and hence
$$S_{m,n}(0) = \sum_{k=-m}^{n} c_k e^{ik\cdot 0} = \sum_{k=-m}^{n} c_k = \sum_{k=-m}^{n} (\bar{c}_{k-1} - \bar{c}_k) = \bar{c}_{-m-1} - \bar{c}_n.$$
3. By Riemann-Lebesgue, $\lim_{n,m\to\infty} S_{m,n}(0) = 0 = f(0)$.
4. Let $a, f(a)$ be any pair of real numbers, and define
$$g(x) = f(x + a) - f(a).$$
Then $g(0) = 0$, $g'(0)$ exists, and $S^g_{m,n}(x) = S^f_{m,n}(x + a) - f(a)$. By 3.
$$|S^f_{m,n}(a) - f(a)| = |S^g_{m,n}(0)| \to 0 \text{ as } m, n \to \infty.$$