Complex Conjugate Numbers

The complex conjugate $\bar{z}$ of a complex number $z = x + iy$ is defined by

$$\bar{z} = x - iy.$$

It is obtained geometrically by reflecting the point z in the real axis. Figure 322 shows this for $z = 5 + 2i$ and its conjugate $\bar{z} = 5 - 2i$.

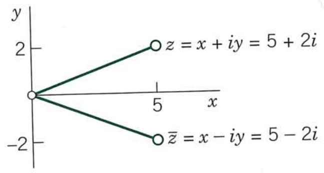

Fig. 322. Complex conjugate numbers

The complex conjugate is important because it permits us to switch from complex to real. Indeed, by multiplication, $z\bar{z} = x^2 + y^2$ (verify!). By addition and subtraction, $z + \bar{z} = 2x$, $z - \bar{z} = 2iy$. We thus obtain for the real part x and the imaginary part y (not iy!) of $z = x + iy$ the important formulas

$$\text{Re } z = x = \tfrac{1}{2}(z + \bar{z}), \qquad \text{Im } z = y = \frac{1}{2i}(z - \bar{z}). \tag{8}$$

If z is real, $z = x$, then $\bar{z} = z$ by the definition of $\bar{z}$, and conversely. Working with conjugates is easy, since we have

$$\overline{(z_1 + z_2)} = \bar{z}_1 + \bar{z}_2, \qquad \overline{(z_1 - z_2)} = \bar{z}_1 - \bar{z}_2,$$
$$\overline{(z_1 z_2)} = \bar{z}_1 \bar{z}_2, \qquad \overline{\left(\frac{z_1}{z_2}\right)} = \frac{\bar{z}_1}{\bar{z}_2}. \tag{9}$$

EXAMPLE 3 Illustration of (8) and (9)

Let $z_1 = 4 + 3i$ and $z_2 = 2 + 5i$. Then by (8),

$$\text{Im } z_1 = \frac{1}{2i}[(4 + 3i) - (4 - 3i)] = \frac{3i + 3i}{2i} = 3.$$

Also, the multiplication formula in (9) is verified by

$$\overline{(z_1 z_2)} = \overline{(4 + 3i)(2 + 5i)} = \overline{(-7 + 26i)} = -7 - 26i,$$
$$\bar{z}_1 \bar{z}_2 = (4 - 3i)(2 - 5i) = -7 - 26i.$$ ■

PROBLEM SET 13.1

1. **Powers of i.** Show that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i, \cdots$ and $1/i = -i$, $1/i^2 = -1$, $1/i^3 = i, \cdots$.
2. **Rotation.** Multiplication by i is geometrically a counterclockwise rotation through $\pi/2$ (90°). Verify this by graphing z and iz and the angle of rotation for $z = 1 + i$, $z = -1 + 2i$, $z = 4 - 3i$.
3. **Division.** Verify the calculation in (7). Apply (7) to $(26 - 18i)/(6 - 2i)$.

4. **Law for conjugates.** Verify (9) for $z_1 = -11 + 10i$, $z_2 = -1 + 4i$.
5. **Pure imaginary number.** Show that $z = x + iy$ is pure imaginary if and only if $\bar{z} = -z$.
6. **Multiplication.** If the product of two complex numbers is zero, show that at least one factor must be zero.
7. **Laws of addition and multiplication.** Derive the following laws for complex numbers from the corresponding laws for real numbers.

$$z_1 + z_2 = z_2 + z_1, \quad z_1 z_2 = z_2 z_1 \quad \textit{(Commutative laws)}$$
$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),$$
$$(z_1 z_2) z_3 = z_1 (z_2 z_3) \quad \textit{(Associative laws)}$$
$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3 \quad \textit{(Distributive law)}$$
$$0 + z = z + 0 = z,$$
$$z + (-z) = (-z) + z = 0, \qquad z \cdot 1 = z.$$

8–15 COMPLEX ARITHMETIC

Let $z_1 = -2 + 5i$, $z_2 = 3 - i$. Showing the details of your work, find, in the form $x + iy$:

8. $z_1 z_2$, $\overline{(z_1 z_2)}$
9. $\text{Re}(z_1^2)$, $(\text{Re } z_1)^2$
10. $\text{Re}(1/z_2^2)$, $1/\text{Re}(z_2^2)$
11. $(z_1 - z_2)^2/16$, $(z_1/4 - z_2/4)^2$
12. z_1/z_2, z_2/z_1
13. $(z_1 + z_2)(z_1 - z_2)$, $z_1^2 - z_2^2$
14. $\bar{z}_1/\bar{z}_2$, $\overline{(z_1/z_2)}$
15. $4(z_1 + z_2)/(z_1 - z_2)$

16–20 Let $z = x + iy$. Showing details, find, in terms of x and y:

16. $\text{Im}(1/z)$, $\text{Im}(1/z^2)$
17. $\text{Re } z^4 - (\text{Re } z^2)^2$
18. $\text{Re}[(1 + i)^{16} z^2]$
19. $\text{Re}(z/\bar{z})$, $\text{Im}(z/\bar{z})$
20. $\text{Im}(1/\bar{z}^2)$

13.2 Polar Form of Complex Numbers. Powers and Roots

We gain further insight into the arithmetic operations of complex numbers if, in addition to the xy-coordinates in the complex plane, we also employ the usual polar coordinates r, θ defined by

$$x = r\cos\theta, \qquad y = r\sin\theta. \tag{1}$$

We see that then $z = x + iy$ takes the so-called **polar form**

$$z = r(\cos\theta + i\sin\theta). \tag{2}$$

r is called the **absolute value** or **modulus** of z and is denoted by $|z|$. Hence

$$|z| = r = \sqrt{x^2 + y^2} = \sqrt{z\bar{z}}. \tag{3}$$

Geometrically, $|z|$ is the distance of the point z from the origin (Fig. 323). Similarly, $|z_1 - z_2|$ is the distance between z_1 and z_2 (Fig. 324).

θ is called the **argument** of z and is denoted by arg z. Thus $\theta = \arg z$ and (Fig. 323)

$$\tan\theta = \frac{y}{x} \qquad (z \neq 0). \tag{4}$$

Geometrically, θ is the directed angle from the positive x-axis to OP in Fig. 323. Here, as in calculus, all ***angles are measured in radians and positive in the counterclockwise sense.***

If ω denotes the value corresponding to $k = 1$ in (16), then the n values of $\sqrt[n]{1}$ can be written as

$$1, \omega, \omega^2, \cdots, \omega^{n-1}.$$

More generally, if w_1 is any nth root of an arbitrary complex number $z\ (\neq 0)$, then the n values of $\sqrt[n]{z}$ in (15) are

$$w_1, \quad w_1\omega, \quad w_1\omega^2, \quad \cdots, \quad w_1\omega^{n-1} \tag{17}$$

because multiplying w_1 by ω^k corresponds to increasing the argument of w_1 by $2k\pi/n$. Formula (17) motivates the introduction of roots of unity and shows their usefulness.

PROBLEM SET 13.2

1–8 POLAR FORM

Represent in polar form and graph in the complex plane as in Fig. 325. Do these problems very carefully because polar forms will be needed frequently. Show the details.

1. $1 + i$
2. $-2 + 2i$
3. $2i, \quad -2i$
4. -4
5. $\dfrac{\sqrt{2} + i/3}{-\sqrt{8} - 2i/3}$
6. $\dfrac{\sqrt{5} - 10i}{-\frac{1}{2}\sqrt{5} + 5i}$
7. $1 + \frac{1}{2}\pi i$
8. $\dfrac{7 + 4i}{3 - 2i}$

9–14 PRINCIPAL ARGUMENT

Determine the principal value of the argument and graph it as in Fig. 325.

9. $1 - i$
10. $-5, \quad -5 - i, \quad -5 + i$
11. $\sqrt{3} \pm i$
12. $-\pi - \pi i$
13. $(1 - i)^{20}$
14. $-1 + 0.1i, \quad -1 - 0.1i$

15–18 CONVERSION TO $x + iy$

Graph in the complex plane and represent in the form $x + iy$:

15. $4\,(\cos\frac{\pi}{2} - i\sin\frac{\pi}{2})$
16. $6\,(\cos\frac{1}{3}\pi + i\sin\frac{1}{3}\pi)$
17. $\sqrt{8}\,(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4})$
18. $\sqrt{50}\,(\cos\frac{3}{4}\pi + i\sin\frac{3}{4}\pi)$

ROOTS

19. **CAS PROJECT. Roots of Unity and Their Graphs.** Write a program for calculating these roots and for graphing them as points on the unit circle. Apply the program to $z^n = 1$ with $n = 2, 3, \cdots, 10$. Then extend the program to one for arbitrary roots, using an idea near the end of the text, and apply the program to examples of your choice.

20. **TEAM PROJECT. Square Root. (a)** Show that $w = \sqrt{z}$ has the values

$$\begin{aligned} w_1 &= \sqrt{r}\left[\cos\frac{\theta}{2} + i\sin\frac{\theta}{2}\right], \\ w_2 &= \sqrt{r}\left[\cos\left(\frac{\theta}{2} + \pi\right) + i\sin\left(\frac{\theta}{2} + \pi\right)\right] \\ &= -w_1. \end{aligned} \tag{18}$$

(b) Obtain from (18) the often more practical formula

$$\sqrt{z} = \pm[\sqrt{\tfrac{1}{2}(|z| + x)} + (\text{sign } y)i\sqrt{\tfrac{1}{2}(|z| - x)}] \tag{19}$$

where sign $y = 1$ if $y \geqq 0$, sign $y = -1$ if $y < 0$, and all square roots of positive numbers are taken with positive sign. *Hint:* Use (10) in App. A3.1 with $x = \theta/2$.

(c) Find the square roots of $-14i, -9 - 40i$, and $1 + \sqrt{48}i$ by both (18) and (19) and comment on the work involved.

(d) Do some further examples of your own and apply a method of checking your results.

21–27 ROOTS

Find and graph all roots in the complex plane.

21. $\sqrt[3]{1 - i}$
22. $\sqrt[3]{3 + 4i}$
23. $\sqrt[3]{343}$
24. $\sqrt[4]{-4}$
25. $\sqrt[4]{i}$
26. $\sqrt[8]{1}$
27. $\sqrt[5]{-1}$

28–31 EQUATIONS

Solve and graph the solutions. Show details.

28. $z^2 - (6 - 2i)z + 17 - 6i = 0$
29. $z^2 - z + 1 + i = 0$
30. $z^4 + 324 = 0$. Using the solutions, factor $z^4 + 324$ into quadratic factors with *real* coefficients.
31. $z^4 - 6iz^2 + 16 = 0$

32–35 INEQUALITIES AND EQUALITY

32. **Triangle inequality.** Verify (6) for $z_1 = 3 + i$, $z_2 = -2 + 4i$
33. **Triangle inequality.** Prove (6).
34. **Re and Im.** Prove $|\text{Re}\, z| \leqq |z|$, $\quad |\text{Im}\, z| \leqq |z|$.
35. **Parallelogram equality.** Prove and explain the name

$$|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2\,(|z_1|^2 + |z_2|^2).$$

13.3 Derivative. Analytic Function

Just as the study of calculus or real analysis required concepts such as domain, neighborhood, function, limit, continuity, derivative, etc., so does the study of complex analysis. Since the functions live in the complex plane, the concepts are slightly more difficult or *different* from those in real analysis. This section can be seen as a reference section where many of the concepts needed for the rest of Part D are introduced.

Circles and Disks. Half-Planes

The **unit circle** $|z| = 1$ (Fig. 330) has already occurred in Sec. 13.2. Figure 331 shows a general circle of radius ρ and center a. Its equation is

$$|z - a| = \rho$$

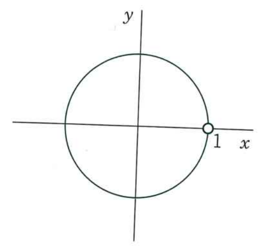

Fig. 330. Unit circle

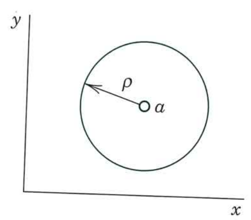

Fig. 331. Circle in the complex plane

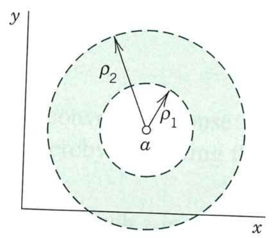

Fig. 332. Annulus in the complex plane

because it is the set of all z whose distance $|z - a|$ from the center a equals ρ. Accordingly, its interior (**"open circular disk"**) is given by $|z - a| < \rho$, its interior plus the circle itself (**"closed circular disk"**) by $|z - a| \leqq \rho$, and its exterior by $|z - a| > \rho$. As an example, sketch this for $a = 1 + i$ and $\rho = 2$, to make sure that you understand these inequalities.

An open circular disk $|z - a| < \rho$ is also called a **neighborhood** of a or, more precisely, a *ρ-neighborhood* of a. And a has infinitely many of them, one for each value of $\rho\ (> 0)$, and a is a point of each of them, by definition!

In modern literature *any set* containing a ρ-neighborhood of a is also called a *neighborhood* of a.

Figure 332 shows an **open annulus** (circular ring) $\rho_1 < |z - a| < \rho_2$, which we shall need later. This is the set of all z whose distance $|z - a|$ from a is greater than ρ_1 but less than ρ_2. Similarly, the **closed annulus** $\rho_1 \leqq |z - a| \leqq \rho_2$ includes the two circles.

Half-Planes. By the (open) *upper* **half-plane** we mean the set of all points $z = x + iy$ such that $y > 0$. Similarly, the condition $y < 0$ defines the *lower half-plane*, $x > 0$ the *right half-plane*, and $x < 0$ the *left half-plane*.

EXAMPLE 5 **Polynomials, Rational Functions**

The nonnegative integer powers $1, z, z^2, \cdots$ are analytic in the entire complex plane, and so are **polynomials**, that is, functions of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + \cdots + c_n z^n$$

where $c_0, \cdots, c_n$ are complex constants.

The quotient of two polynomials $g(z)$ and $h(z)$,

$$f(z) = \frac{g(z)}{h(z)},$$

is called a **rational function**. This f is analytic except at the points where $h(z) = 0$; here we assume that common factors of g and h have been canceled.

Many further analytic functions will be considered in the next sections and chapters. ■

The concepts discussed in this section extend familiar concepts of calculus. Most important is the concept of an analytic function, the exclusive concern of complex analysis. Although many simple functions are not analytic, the large variety of remaining functions will yield a most beautiful branch of mathematics that is very useful in engineering and physics.

PROBLEM SET 13.3

1–8 **REGIONS OF PRACTICAL INTEREST**

Determine and sketch or graph the sets in the complex plane given by

1. $|z + 1 - 2i| \leqq \frac{1}{4}$
2. $0 < |z| < 1$
3. $\frac{\pi}{2} < |z - 1 + 2i| < \pi$
4. $-\pi < \text{Im}\, z < \pi$
5. $|\arg z| < \frac{\pi}{3}$
6. $\text{Re}\,(1/z) < 1$
7. $\text{Re}\, z \leqq -1$
8. $|z + i| \geqq |z - i|$
9. **WRITING PROJECT. Sets in the Complex Plane.** Write a report by formulating the corresponding portions of the text in your own words and illustrating them with examples of your own.

COMPLEX FUNCTIONS AND THEIR DERIVATIVES

10–12 **Function Values.** Find Re f, and Im f and their values at the given point z.

10. $f(z) = 5z^2 - 12z + 3 + 2i$ at $4 - 3i$
11. $f(z) = 1/(1 + z)$ at $1 - i$
12. $f(z) = (z - 1)/(z + 1)$ at $2i$
13. **CAS PROJECT. Graphing Functions.** Find and graph Re f, Im f, and $|f|$ as surfaces over the z-plane. Also graph the two families of curves $\text{Re}\, f(z) = \text{const}$ and $\text{Im}\, f(z) = \text{const}$ in the same figure, and the curves $|f(z)| = \text{const}$ in another figure, where (**a**) $f(z) = z^2$, (**b**) $f(z) = 1/z$, (c) $f(z) = z^4$.

14–17 **Continuity.** Find out, and give reason, whether $f(z)$ is continuous at $z = 0$ if $f(0) = 0$ and for $z \neq 0$ the function f is equal to:

14. $(\text{Re}\, z^2)/|z|$
15. $|z|^2\, \text{Im}\,(1/z)$
16. $(\text{Im}\, z^2)/|z|^2$
17. $(\text{Re}\, z)/(1 - |z|)$

18–23 **Differentiation.** Find the value of the derivative of

18. $(z - i)/(z + i)$ at i
19. $(z - 2i)^3$ at $= 5 + 2i$
20. $(1.5z + 2i)/(3iz - 4)$ at any z. Explain the result.
21. $i(1 - z)^n$ at 0
22. $(iz^3 + 3z^2)^3$ at $2i$
23. $z^3/(z - i)^3$ at $-i$
24. **TEAM PROJECT. Limit, Continuity, Derivative**

 (**a**) **Limit.** Prove that (1) is equivalent to the pair of relations

$$\lim_{z \to z_0} \text{Re}\, f(z) = \text{Re}\, l, \qquad \lim_{z \to z_0} \text{Im}\, f(z) = \text{Im}\, l.$$

 (**b**) **Limit.** If $\lim_{z \to z_0} f(x)$ exists, show that this limit is unique.

 (**c**) **Continuity.** If $z_1, z_2, \cdots$ are complex numbers for which $\lim_{n \to \infty} z_n = a$, and if $f(z)$ is continuous at $z = a$, show that $\lim_{n \to \infty} f(z_n) = f(a)$.

 (**d**) **Continuity.** If $f(z)$ is differentiable at z_0, show that $f(z)$ is continuous at z_0.

 (**e**) **Differentiability.** Show that $f(z) = \text{Re}\, z = x$ is not differentiable at any z. Can you find other such functions?

 (**f**) **Differentiability.** Show that $f(z) = |z|^2$ is differentiable only at $z = 0$; hence it is nowhere analytic.

25. **WRITING PROJECT. Comparison with Calculus.** Summarize the second part of this section beginning with *Complex Function*, and indicate what is conceptually analogous to calculus and what is not.

13.4 Cauchy–Riemann Equations. Laplace's Equation

As we saw in the last section, to do complex analysis (i.e., "calculus in the complex") on any complex function, we require that function to be *analytic on some domain* that is differentiable in that domain.

The Cauchy–Riemann equations are the most important equations in this chapter and one of the pillars on which complex analysis rests. They provide a criterion (a test) for the analyticity of a complex function

$$w = f(z) = u(x, y) + iv(x, y).$$

Roughly, f is analytic in a domain D if and only if the first partial derivatives of u and v satisfy the two **Cauchy–Riemann equations**[4]

$$u_x = v_y, \qquad u_y = -v_x \tag{1}$$

everywhere in D; here $u_x = \partial u/\partial x$ and $u_y = \partial u/\partial y$ (and similarly for v) are the usual notations for partial derivatives. The precise formulation of this statement is given in Theorems 1 and 2.

Example: $f(z) = z^2 = x^2 - y^2 + 2ixy$ is analytic for all z (see Example 3 in Sec. 13.3), and $u = x^2 - y^2$ and $v = 2xy$ satisfy (1), namely, $u_x = 2x = v_y$ as well as $u_y = -2y = -v_x$. More examples will follow.

THEOREM 1 **Cauchy–Riemann Equations**

Let $f(z) = u(x, y) + iv(x, y)$ be defined and continuous in some neighborhood of a point $z = x + iy$ and differentiable at z itself. Then, at that point, the first-order partial derivatives of u and v exist and satisfy the Cauchy–Riemann equations (1).

Hence, if $f(z)$ is analytic in a domain D, those partial derivatives exist and satisfy (1) at all points of D.

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[4]The French mathematician AUGUSTIN-LOUIS CAUCHY (see Sec. 2.5) and the German mathematicians BERNHARD RIEMANN (1826–1866) and KARL WEIERSTRASS (1815–1897; see also Sec. 15.5) are the founders of complex analysis. Riemann received his Ph.D. (in 1851) under Gauss (Sec. 5.4) at Göttingen, where he also taught until he died, when he was only 39 years old. He introduced the concept of the integral as it is used in basic calculus courses, and made important contributions to differential equations, number theory, and mathematical physics. He also developed the so-called Riemannian geometry, which is the mathematical foundation of Einstein's theory of relativity; see Ref. [GenRef9] in App. 1.