

Miscellaneous

- **Heaviside function** $u(t) = \begin{cases} 1, & t \geq 0 \\ 0, & t < 0 \end{cases}$, $u(t-a) = \begin{cases} 1, & t \geq a \\ 0, & t < a \end{cases}$
- **Dirac Delta function** $\delta(t-a)$ is zero except at $t=a$, $\int_{-\infty}^{\infty} \delta(t-a)dt = 1$, and $\int_{-\infty}^{\infty} g(t)\delta(t-a)dt = g(a)$ for any continuous function g .
- **Convolution**

For functions defined on the real line:

$$f * g(x) = \int_{-\infty}^{\infty} f(y)g(x-y)dy = \int_{-\infty}^{\infty} f(x-y)g(y)dy, \quad x \in \mathbb{R}.$$

For functions defined only on the positive half-axis:

$$f * g(x) = \int_0^x f(y)g(x-y)dy, \quad x > 0.$$

Laplace transform

- Definition: $\mathcal{L}[f] = F(s) = \int_0^{\infty} f(t)e^{-st}dt$

General formulas	$f(t)$	$F(s)$
	1	$\frac{1}{s}$
$\mathcal{L}[e^{at}f(t)] = F(s-a)$	$t^n, n = 1, 2, \dots$	$\frac{n!}{s^{n+1}}$
$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$	e^{at}	$\frac{1}{s-a}$
$\mathcal{L}[f''] = s^2\mathcal{L}[f] - sf(0) - f'(0)$	$t^n e^{at}, n = 1, 2, \dots$	$\frac{n!}{(s-a)^{n+1}}$
$\mathcal{L}\left[\int_0^t f(\tau)d\tau\right] = \frac{1}{s}\mathcal{L}[f]$	$\cos bt$	$\frac{s}{s^2+b^2}$
$\mathcal{L}[f * g] = \mathcal{L}[f]\mathcal{L}[g]$	$\sin bt$	$\frac{b}{s^2+b^2}$
$\mathcal{L}[f(t-c)u(t-c)] = e^{-cs}F(s), c > 0$	$e^{at} \cos bt$	$\frac{s-a}{(s-a)^2+b^2}$
$\mathcal{L}[tf(t)] = -F'(s)$	$e^{at} \sin bt$	$\frac{b}{(s-a)^2+b^2}$
$\mathcal{L}\left[\frac{f(t)}{t}\right] = \int_s^{\infty} F(\sigma)d\sigma$	$u(t-c), c > 0$	$\frac{e^{-cs}}{s}$
	$\delta(t-c), c > 0$	e^{-cs}

Fourier series and Fourier transform

- $2L$ -periodic functions, real and complex form

$$f(x) \sim a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi x}{L} + b_n \sin \frac{n\pi x}{L} \right) \sim \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$$

where

$$a_0 = \frac{1}{2L} \int_{-L}^L f(x) dx, \quad a_n = \frac{1}{L} \int_{-L}^L f(x) \cos \frac{n\pi x}{L} dx, \quad b_n = \frac{1}{L} \int_{-L}^L f(x) \sin \frac{n\pi x}{L} dx$$

$$c_n = \frac{1}{2L} \int_{-L}^L f(x) e^{-\frac{in\pi x}{L}} dx$$

- Functions defined on the whole real line (need not be periodic)

$$\hat{f}(w) = \mathcal{F}[f](w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx,$$

$$f(x) = \mathcal{F}^{-1}[\hat{f}](x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw.$$

- Parseval's identities

$$\frac{1}{2L} \int_{-L}^L |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2, \quad \int_{-\infty}^{\infty} |f(x)|^2 dx = \int_{-\infty}^{\infty} |\hat{f}(w)|^2 dw$$

General formulas		$f(x)$	$\hat{f}(w)$
$\widehat{f'(x)} = iw\hat{f}(w)$		$\delta(x - a)$	$\frac{1}{\sqrt{2\pi}} e^{-iaw}$
$\widehat{f''(x)} = -w^2\hat{f}(w)$		$\begin{cases} 1, & -b \leq x \leq b \\ 0, & x > b \end{cases}$	$\sqrt{\frac{2}{\pi}} \frac{\sin bw}{w}$
$\widehat{f(x - a)} = e^{-iaw}\hat{f}(w)$		$e^{-ax}u(x)$	$\frac{1}{\sqrt{2\pi}(a + iw)}$
$\hat{f}(w - b) = e^{ibx}\widehat{f(x)}$		$\frac{1}{x^2 + a^2}$	$\sqrt{\frac{\pi}{2}} \frac{e^{-a w }}{a}$
$\widehat{f * g} = \sqrt{2\pi}\hat{f}\hat{g}$		e^{-ax^2}	$\frac{1}{\sqrt{2a}} e^{-w^2/(4a)}$

Complex numbers and analytic functions

- $e^{x+iy} = e^x(\cos y + i \sin y)$
- $\cos z = \frac{e^{iz} + e^{-iz}}{2}$, $\sin z = \frac{e^{iz} - e^{-iz}}{2i}$, $\cosh z = \frac{e^z + e^{-z}}{2}$, $\sinh z = \frac{e^z - e^{-z}}{2}$
- Taylor and Laurent series of an analytic function

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad a_n = \frac{f^{(n)}(z_0)}{n!} = \frac{1}{2\pi i} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} dz,$$

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z - z_0)^n}, \quad b_n = \frac{1}{2\pi i} \oint_C f(z) (z - z_0)^{n-1} dz$$

Some useful integrals

$$\int x \sin ax \, dx = \frac{1}{a^2} \sin ax - \frac{x}{a} \cos ax + C$$

$$\int x \cos ax \, dx = \frac{1}{a^2} \cos ax + \frac{x}{a} \sin ax + C$$

$$\int x^2 \sin ax \, dx = \frac{2}{a^2} x \sin ax + \frac{2-a^2 x^2}{a^3} \cos ax + C$$

$$\int x^2 \cos ax \, dx = \frac{2}{a^2} x \cos ax - \frac{2-a^2 x^2}{a^3} \sin ax + C$$

$$\int e^{ax} \sin bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \sin bx - b \cos bx) + C$$

$$\int e^{ax} \cos bx \, dx = \frac{e^{ax}}{a^2 + b^2} (a \cos bx + b \sin bx) + C$$

$$\int_{-\infty}^{\infty} e^{-ax^2} \, dx = \sqrt{\frac{\pi}{a}}, \quad a > 0$$

Some trigonometric identities

$$\cos(a \pm b) = \cos a \cos b \mp \sin a \sin b$$

$$\sin(a \pm b) = \sin a \cos b \pm \cos a \sin b$$

Some important series

- $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$ for $|x| < 1$, $\sum_{n=0}^{\infty} x^n$ diverges for $|x| \geq 1$.
- $\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$ for $x \in \mathbb{R}$.